6.5 Intersections of Lines in 2-Space and 3-Space Worksheet with solutions

This document was composed and formatted by Annie Bernatchez.

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1) Solve each linear system in two-space.

a) $3x - 5y = -9,\ 4x + 5y = 23$

Add: $7x = 14 \rightarrow x = 2$

$3(2) - 5y = -9 \rightarrow -5y = -15 \rightarrow y = 3$

The solution is $(2, 3)$.

b) $\ell_1 : [x, y] = [5, 4] + s[-3, 1],\ \ell_2 : [x, y] = [2, 2] + t[2, -1]$

$\ell_1 : \begin{cases} x = 5 - 3s \\ y = 4 + s \end{cases}$ $\ell_2 : \begin{cases} x = 2 + 2t \\ y = 2 - t \end{cases}$

$5 - 3s = 2 + 2t \rightarrow 3 = 2t + 3s$

$4 + s = 2 - t \rightarrow 2 = -t - s$

$2 \times (2 = -t - s) \rightarrow 4 = -2t - 2s$

Add to $3 = 2t + 3s$: $\rightarrow 7 = s$

$3 = 2t + 3(7) \rightarrow t = -9$

$\ell_1 : x = 5 - 3(7) = -16,\ y = 4 + 7 = 11$

Check in ℓ_2: $x = 2 + 2(-9) = -16,\ y = 2 - (-9) = 11$

The solution is $(-16, 11)$.

2) Determine if the parallel lines in each pair are distinct or coincident.

a) $\ell_1 : [x, y, z] = [5, -2, -8] + s[-3, 2, 5]$

$\ell_2 : [x, y, z] = [-4, 0, 2] + t[-3, 2, 5]$

Check if point $(-4, 0, 2)$ is on ℓ_1:

$\ell_1 : \begin{cases} x = 5 - 3s \\ y = -2 + 2s \\ z = -8 + 5s \end{cases}$

$-4 = 5 - 3s \rightarrow s = 3$

$0 = -2 + 2s \rightarrow s = 1$

$2 = -8 + 5s \rightarrow s = 2$

Since the values of s are not equal, the point is not on ℓ_1. Therefore, the lines are parallel and distinct.

b) $\ell_1 : [x, y, z] = [-3, 0, -6] + s[-3, 0, -6]$

$\ell_2 : [x, y, z] = [9, 1, 18] + t[9, 0, 18]$

Check if point $(-3, 0, -6)$ is on ℓ_2:

$\ell_2 : \begin{cases} x = 9 + 9t \\ y = 1 \\ z = 18 + 18t \end{cases}$

$-3 = 9 + 9t \rightarrow t = -\frac{4}{3}$

$0 = 1 \rightarrow$ Not possible

Since $y = 1$ can never equal 0, the point is not on ℓ_2. Therefore, the lines are parallel and distinct.

3) Triangle ABC is formed from the intersections of the three lines represented by these equations. Find the length of each side of $\triangle ABC$.

$\ell_1 : [x, y] = [1, -3] + t[0, 1]$ $\ell_2 : [x, y] = [2, 4] + s[-1, 6]$ $\ell_3 : [x, y] = [2, 3] + r[1, 7]$

$\ell_1 : \begin{cases} x = 1 \\ y = -3 + t \end{cases}$ $\ell_2 : \begin{cases} x = 2 - s \\ y = 4 + 6s \end{cases}$ $\ell_3 : \begin{cases} x = 2 + r \\ y = 3 + 7r \end{cases}$

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Intersection of ℓ_1 and ℓ_2:

$1 = 2 - s \rightarrow s = 1$

$x = 2 - 1 = 1,\ \ y = 4 + 6(1) = 10$

$P_1 = (1,\ 10)$

Intersection of ℓ_2 and ℓ_3:

$2 - s = 2 + r \rightarrow 0 = s + r$

$4 + 6s = 3 + 7r \rightarrow 1 = -6s + 7r$

$6\times(0 = s + r) \rightarrow 0 = 6s + 6r$

Add: $1 = 13r \rightarrow r = \frac{1}{13}$

$x = 2 + \frac{1}{13} = \frac{27}{13},\ \ y = 3 + 7\left(\frac{1}{13}\right) = \frac{46}{13}$

$P_2 = \left(\frac{27}{13},\ \frac{46}{13}\right)$

Intersection of ℓ_1 and ℓ_3:

$1 = 2 + r \rightarrow r = -1$

$x = 2 + (-1) = 1,\ \ y = 3 + 7(-1) = -4$

$P_3 = (1,\ -4)$

$\overrightarrow{P_1P_2} = \left[\frac{14}{13},\ -\frac{84}{13}\right]$

$\overrightarrow{P_1P_3} = [0,\ -14]$

$\overrightarrow{P_2P_3} = \left[-\frac{14}{13},\ -\frac{98}{13}\right]$

$|\overrightarrow{P_1P_2}| = \sqrt{7252}/13 = \frac{14\sqrt{37}}{13}$

$|\overrightarrow{P_1P_2}| \cong 6.55$

$|\overrightarrow{P_1P_3}| = \sqrt{196} = 14$

$|\overrightarrow{P_1P_3}| = 14$

$|\overrightarrow{P_2P_3}| = \sqrt{9800}/13 = \frac{70\sqrt{2}}{13}$

$|\overrightarrow{P_2P_3}| \cong 7.61$

4) Parallelogram ABCD has vertices $A(-1,\ -4)$, $B(1,\ -3)$, $C(6,\ -6)$, and $D(4,\ -7)$. Find the vector equations of its diagonals and the point of intersection of the diagonals.

$\overrightarrow{AC} = [6 - (-1),\ -6 - (-4)] = [7,\ -2]$

$\overrightarrow{BD} = [4 - 1,\ -7 - (-3)] = [3,\ -4]$

$\ell_1 : [x,\ y] = [-1,\ -4] + t[7,\ -2]$

$\ell_2 : [x,\ y] = [1,\ -3] + s[3,\ -4]$

$\ell_1 : \begin{cases} x = -1 + 7t \\ y = -4 - 2t \end{cases}\ \ \ell_2 : \begin{cases} x = 1 + 3s \\ y = -3 - 4s \end{cases}$

$-1 + 7t = 1 + 3s \rightarrow -2 = -7t + 3s$

$-4 - 2t = -3 - 4s \rightarrow -1 = 2t - 4s$

$4\times(-2 = -7t + 3s) \rightarrow -8 = -28t + 12s$

$3\times(-1 = 2t - 4s) \rightarrow -3 = 6t - 12s$

Add: $-11 = -22t \rightarrow t = \frac{1}{2}$

$x = -1 + 7\left(\frac{1}{2}\right) = \frac{5}{2}$

$y = -4 - 2\left(\frac{1}{2}\right) = -5$

The diagonals intersect at $(2.5,\ -5)$.

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5) Determine if the lines in each pair intersect. If so, find the co-ordinates of the point of intersection.

a) $\ell_1 : [x, y, z] = [9, -1, 1] + s[-3, 4, 1]$

$\ell_2 : [x, y, z] = [-3, 11, 5] + t[-3, 4, 1]$

The direction vectors are identical, so the lines are parallel. Check if point $(-3, 11, 5)$ is on ℓ_1:

$-3 = 9 - 3s \rightarrow s = 4$

$11 = -1 + 4s \rightarrow s = 3$

Since the values of s are not equal, the point is not on ℓ_1. Therefore, there are no solutions; the lines are parallel and distinct.

b) $\ell_1 : [x, y, z] = [1, 4, 5] + s[3, 0, -2]$

$\ell_2 : [x, y, z] = [9, 4, -3] + t[3, 0, -2]$

The direction vectors are identical, so the lines are parallel. Check if point $(1, 4, 5)$ is on ℓ_2:

$1 = 9 + 3t \rightarrow t = -\frac{8}{3}$

$5 = -3 - 2t \rightarrow t = -4$

Since the values of t are not equal, the point is not on ℓ_2. Therefore, there are no solutions; the lines are parallel and distinct.

c) $\ell_1 : [x, y, z] = [1, 0, -3] + t[3, 5, 4]$

$\ell_2 : [x, y, z] = [0, -9, -1] + s[-1, 2, -3]$

The direction vectors are not scalar multiples, so the lines are not parallel.

$\ell_1 : \begin{cases} x = 1 + 3t \\ y = 5t \\ z = -3 + 4t \end{cases} \quad \ell_2 : \begin{cases} x = -s \\ y = -9 + 2s \\ z = -1 - 3s \end{cases}$

$1 + 3t = -s \rightarrow 3t + s = -1$

$5t = -9 + 2s \rightarrow 5t - 2s = -9$

$-3 + 4t = -1 - 3s \rightarrow 4t + 3s = 2$

$2 \times (3t + s = -1) \rightarrow 6t + 2s = -2$

Add to $5t - 2s = -9 : \rightarrow 11t = -11 \rightarrow t = -1$

$3(-1) + s = -1 \rightarrow s = 2$

Check in the third equation:

$\text{LS} = 4(-1) + 3(2) = 2$

LS = RS = 2. Therefore, the lines intersect. Sub $t = -1$ into ℓ_1:

$x = 1 + 3(-1) = -2, \; y = 5(-1) = -5, \; z = -3 + 4(-1) = -7$

The point of intersection is $(-2, -5, -7)$.

d) $\ell_1 : [x, y, z] = [4, 7, -1] + s[-2, 1, 2]$

$\ell_2 : [x, y, z] = [1, 3, -1] + t[4, -1, 2]$

The direction vectors are not scalar multiples, so the lines are not parallel.

$\ell_1 : \begin{cases} x = 4 - 2s \\ y = 7 + s \\ z = -1 + 2s \end{cases} \quad \ell_2 : \begin{cases} x = 1 + 4t \\ y = 3 - t \\ z = -1 + 2t \end{cases}$

$4 - 2s = 1 + 4t \rightarrow 3 = 4t + 2s$

$7 + s = 3 - t \rightarrow 4 = -t - s$

$-1 + 2s = -1 + 2t \rightarrow 0 = 2t - 2s$

$2 \times (4 = -t - s) \rightarrow 8 = -2t - 2s$

Add to $3 = 4t + 2s : \rightarrow 11 = 2t \rightarrow t = \frac{11}{2}$

$4 = -\frac{11}{2} - s \rightarrow s = -\frac{19}{2}$

Check in the third equation:

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$\text{RS} = 2\left(\frac{11}{2}\right) - 2\left(-\frac{19}{2}\right) = 11 + 19 = 30$

LS $= 0 \neq$ RS $= 30$. Therefore, there is no solution; the lines are skew.

e) $\ell_1 : [x, y, z] = [-2, 0, -3] + t[5, 1, 3]$

$\ell_2 : [x, y, z] = [5, 8, -6] + s[-1, 2, -3]$

The direction vectors are not scalar multiples, so the lines are not parallel.

$\ell_1 : \begin{cases} x = -2 + 5t \\ y = t \\ z = -3 + 3t \end{cases} \quad \ell_2 : \begin{cases} x = 5 - s \\ y = 8 + 2s \\ z = -6 - 3s \end{cases}$

$-2 + 5t = 5 - s \rightarrow 5t + s = 7$

$t = 8 + 2s \rightarrow t - 2s = 8$

$-3 + 3t = -6 - 3s \rightarrow 3t + 3s = -3$

$2 \times (5t + s = 7) \rightarrow 10t + 2s = 14$

Add to $t - 2s = 8$: $\rightarrow 11t = 22 \rightarrow t = 2$

$5(2) + s = 7 \rightarrow s = -3$

Check in the third equation:

$3(2) + 3(-3) = -3$

They match. Therefore, the lines intersect. Sub $t = 2$ into ℓ_1:

$x = -2 + 5(2) = 8, \ y = 2, \ z = -3 + 3(2) = 3$

The point of intersection is $(8, 2, 3)$.

6) Determine the distance between the skew lines in each pair.

a) $\ell_1 : [x, y, z] = [4, -3, 2] + s[2, 7, -1]$

$\ell_2 : [x, y, z] = [-2, 5, 4] + t[-4, 0, 3]$

$\overrightarrow{P_1P_2} = [-2 - 4, 5 - (-3), 4 - 2] = [-6, 8, 2]$

$\vec{n} = [2, 7, -1] \times [-4, 0, 3]$

$\vec{n} = [7(3) - (-1)(0), (-1)(-4) - 2(3), 2(0) - 7(-4)]$

$\vec{n} = [21, -2, 28]$

$d = \left| \frac{[-6, 8, 2] \cdot [21, -2, 28]}{\sqrt{(21)^2 + (-2)^2 + (28)^2}} \right|$

$d = \left| \frac{-86}{\sqrt{1229}} \right|$

$d \cong 2.45$ units

b) $\ell_1 : [x, y, z] = [-1, 6, 1] + s[-2, 4, 3]$

$\ell_2 : [x, y, z] = [5, 1, 9] + t[3, -2, 4]$

$\overrightarrow{P_1P_2} = [5 - (-1), 1 - 6, 9 - 1] = [6, -5, 8]$

$\vec{n} = [-2, 4, 3] \times [3, -2, 4]$

$\vec{n} = [4(4) - 3(-2), 3(3) - (-2)(4), (-2)(-2) - 4(3)]$

$\vec{n} = [22, 17, -8]$

$d = \left| \frac{[6, -5, 8] \cdot [22, 17, -8]}{\sqrt{(22)^2 + (17)^2 + (-8)^2}} \right|$

$d = \left| \frac{-17}{\sqrt{837}} \right|$

$d \cong 0.59$ units

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