6.6 Intersection of Lines and Planes Lesson with solutions

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Part 1: Intersections of Lines and Planes

Given a line and plane in 3-space, there are three possibilities for the intersection of the line with the plane:

Intersect at 1 Point	Infinite Number of Solutions	No Solutions
		
	The line lies on the plane	The line is parallel and distinct from the plane

Example 1: In each case, determine if the line and the plane intersect. If so, determine the solution.

a) $\pi_1 : 9x + 13y - 2z = 29$

$$\ell_1 : \begin{cases} x = 5 + 2t \\ y = -5 - 5t \\ z = 2 + 3t \end{cases}$$

Sub the parametric equations in to the scalar equation of the plane:

$9(5 + 2t) + 13(-5 - 5t) - 2(2 + 3t) = 29$

$45 + 18t - 65 - 65t - 4 - 6t = 29$

$-53t = 53$

$t = -1$

Since a single value of t was found, the line and plane intersect at 1 point. Sub $t = -1$ in to the parametric equations:

$x = 5 + 2(-1) = 3$

$y = -5 - 5(-1) = 0$

$z = 2 + 3(-1) = -1$

The line and plane intersect at the point $(3, 0, -1)$.

b) $\pi_2 : x + 3y - 4z = 10$

$$\ell_2 : \begin{cases} x = 4 + 6t \\ y = -7 + 2t \\ z = 1 + 3t \end{cases}$$

Sub the parametric equations in to the scalar equation of the plane:

$(4 + 6t) + 3(-7 + 2t) - 4(1 + 3t) = 10$

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$4 + 6t - 21 + 6t - 4 - 12t = 10$

$0t = 31$

There are no values of t that can make this equation true. Therefore, the line and the plane do not intersect. The line is parallel and distinct from the plane.

c) $\pi_3 : 4x - y + 11z = -1$

$\ell_3 : [x, y, z] = [-2, 4, 1] + t[3, 1, -1]$

Sub the parametric equations in to the scalar equation of the plane:

$4(-2 + 3t) - (4 + t) + 11(1 - t) = -1$

$-8 + 12t - 4 - t + 11 - 11t = -1$

$0t = 0$

This equation is true for all values of t. There are infinitely many solutions. Any point on the line is a solution. The line lies on the plane.

Example 2: Without solving, determine if each line intersects the plane.

a) $\pi_1 : 3x - y + z = -6$

$\ell_1 : [x, y, z] = [2, -5, 3] + s[3, 2, 1]$

Check if the plane and line are parallel by comparing the direction vector of the line, $\vec{m} = [3, 2, 1]$ *with the normal vector of the plane* $\vec{n} = [3, -1, 1]$. *If they are perpendicular to each other, then the line and plane must be parallel. Find the dot product to analyze:*

$\vec{m} \cdot \vec{n} = [3, 2, 1] \cdot [3, -1, 1]$

$\vec{m} \cdot \vec{n} = 8$

Since $\vec{m} \cdot \vec{n} \neq 0$, $\vec{m}$ and $\vec{n}$ are not perpendicular. Therefore, the line and plane are NOT parallel and must have one point of intersection.

b) $\pi_2 : 4x - 2z = 11$

$\ell_2 : [x, y, z] = [1, 0, 1] + s[-2, 1, -4]$

Check if the plane and line are parallel by comparing the direction vector of the line, $\vec{m} = [-2, 1, -4]$ *with the normal vector of the plane* $\vec{n} = [4, 0, -2]$. *If they are perpendicular to each other, then the line and plane must be parallel. Find the dot product to analyze:*

$\vec{m} \cdot \vec{n} = [-2, 1, -4] \cdot [4, 0, -2]$

$\vec{m} \cdot \vec{n} = -2(4) + 1(0) + (-4)(-2)$

$\vec{m} \cdot \vec{n} = 0$

Since $\vec{m} \cdot \vec{n} = 0$, $\vec{m}$ *and* $\vec{n}$ *ARE perpendicular. Therefore, the line and plane ARE parallel and either have no solutions (parallel and distinct) or infinitely many solutions (line is on plane). Check if the point* $(1, 0, 1)$ *is on the plane to determine which situation we have:*

$LS = 4(1) - 2(1)$

$RS = 11$

$LS = 2$

$LS \neq RS$, therefore the line does not lie on the plane. There are no solutions to this system. The line and plane are parallel and distinct.

Part 2: Distance from a Point to a Plane

Remember the formula for the magnitude of the projection of $\vec{a}$ on to $\vec{b}$ is $|proj_{\vec{b}} \vec{a}| = \frac{|\vec{a} \cdot \vec{b}|}{|\vec{b}|}$

The shortest distance between a point and a plane is the perpendicular distance. This distance, d, between a point P and a plane is $d = |proj_{\vec{n}} \overrightarrow{PQ}| = \frac{|\overrightarrow{PQ} \cdot \vec{n}|}{|\vec{n}|}$ where Q is ANY point on the plane and $\vec{n}$ is a normal vector to the plane.

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Example 3: Find the distance between the plane $4x + 2y + z - 16 = 0$ and the point $P(10,\ 3,\ -8)$.

Find any point Q on the plane by choosing arbitrary values for x and y:

$4(1) + 2(1) + z - 16 = 0$

$z = 10$

Point $Q(1,\ 1,\ 10)$ is on the plane.

Vector $\overrightarrow{PQ} = [1,\ 1,\ 10] - [10,\ 3,\ -8] = [-9,\ -2,\ 18]$

A normal vector to the plane is $\vec{n} = [4,\ 2,\ 1]$

$$d = \frac{|\overrightarrow{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

$$d = \frac{|[-9,\ -2,\ 18] \cdot [4,\ 2,\ 1]|}{\sqrt{(4)^2 + (2)^2 + (1)^2}}$$

$$d = \frac{22}{\sqrt{21}}$$

$d \cong 4.8$ units

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