5.4 Vectors in 3-Space Lesson with solutions

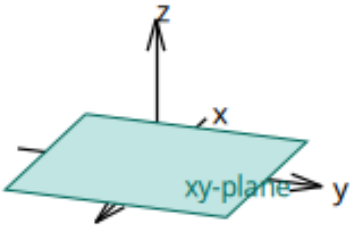
This document was composed and formatted by Annie Bernatchez.

The course material originated from www.jensenmath.ca

Part 1: Plotting Points in 3-Dimensions

www.geogebra.org/3d?lang=en

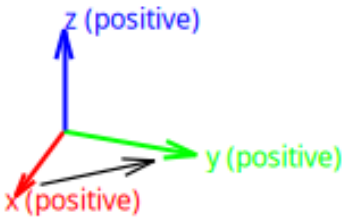
In placing points in 3-Dimensions ($\mathbb{R}^3$), we choose three axes called x-, y-, and z-axis. Each axis is perpendicular. Each point is written using ordered triples (x, y, z).



There are several ways to choose the orientation of the positive axes, but we will use what is called the right-handed system. If we imagine ourselves looking down the positive z-axis onto the xy plane so that, when the positive x-axis is rotated 90° counterclockwise it becomes coincident with the positive y-axis, then this is called the right-handed system.

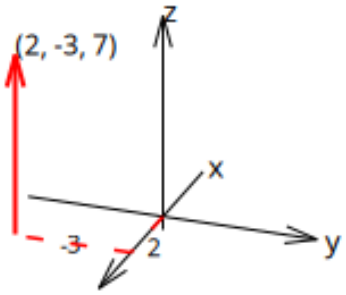
If you curl the fingers of your right hand from the positive x-axis to the positive y-axis your thumb will point along the positive z-axis.

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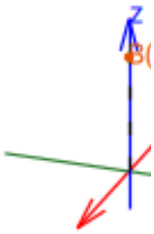
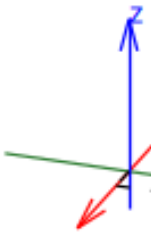
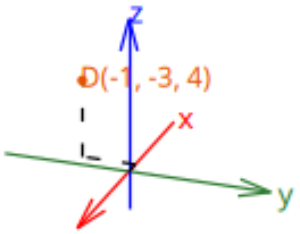
curl right hand from
+x to +y, thumb
points to +z

To plot the point of (2, -3, 7), start at the origin. Move two units along the positive x-axis, then 3 units parallel to the negative y-axis, and then 7 units parallel to the positive z-axis.



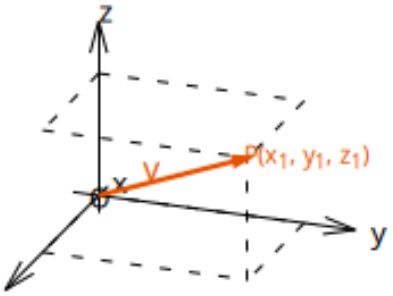
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Example 1: Plot the following points in $\mathbb{R}^3$

a) $A(2,\ 6,\ 1)$	b) $B(0,\ 0,\ 6)$	c) $C(2,\ 3,\ 0)$	d) $D(-1,\ -3,\ 4)$
			

Part 2: 3-D Cartesian Vectors

Let $\vec{v}$ represent a vector in space. If $\vec{v}$ is translated so that its tail is at the origin, O, then its tip will be at some point $P(x_1,\ y_1,\ z_1)$. Then $\vec{v}$ is the position vector of the point P, and $\vec{v} = \overrightarrow{OP} = [x_1,\ y_1,\ z_1]$.



Unit Vectors in $\mathbb{R}^3$

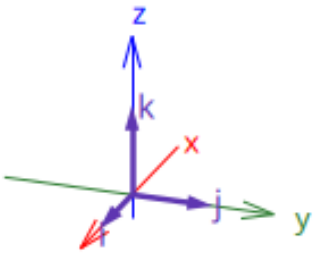
Unit vectors all have a magnitude of 1 and lie along the axes. In 3-Dimensions, there are 3 unit vectors:

x-axis is $\vec{i} = [1,\ 0,\ 0]$

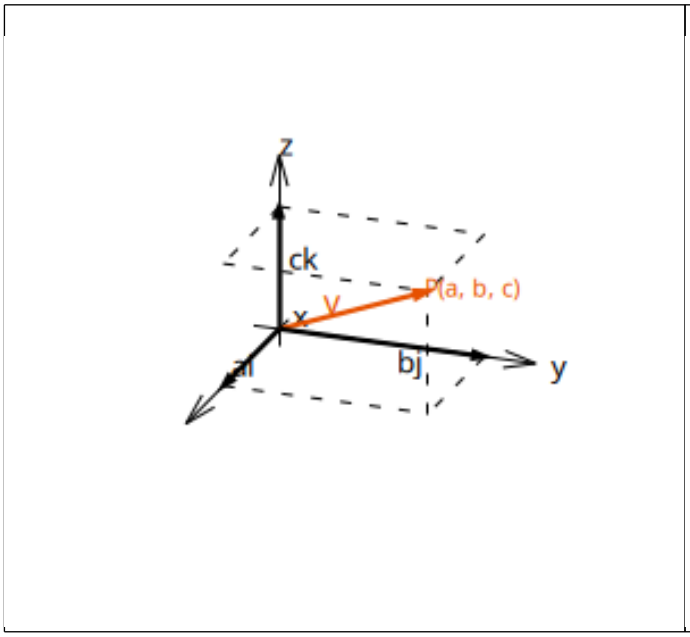
y-axis is $\vec{j} = [0,\ 1,\ 0]$

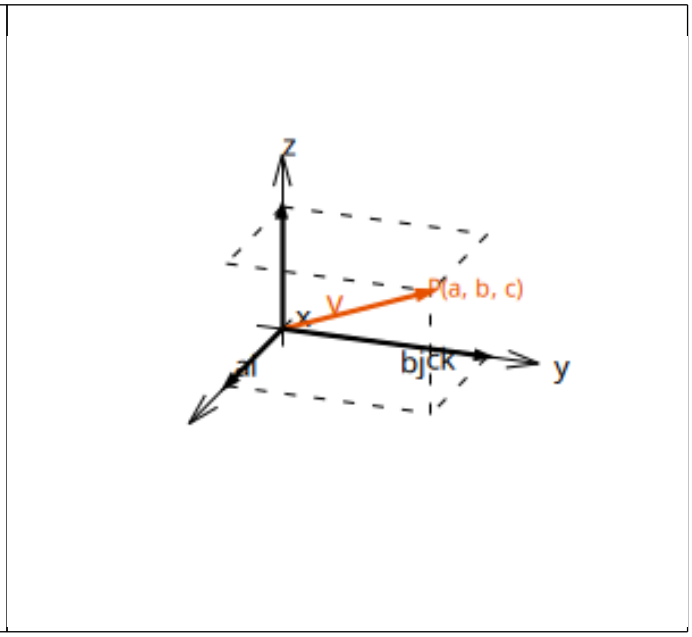
z-axis is $\vec{k} = [0,\ 0,\ 1]$

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3-D vectors can be written as the sum of multiples of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

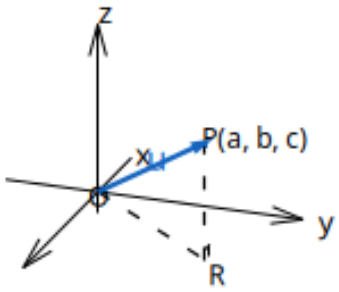




$\vec{v} = [a, b, c] = [a, 0, 0] + [0, b, 0] + [0, 0, c] = a\vec{i} + b\vec{j} + c\vec{k}$

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Part 3: Magnitude of Vectors in $\mathbb{R}^3$



$|\vec{u}|^2 = |\overrightarrow{OP}|^2$

$|\vec{u}|^2 = |\overrightarrow{OR}|^2 + |\overrightarrow{RP}|^2$

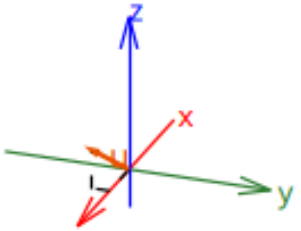
$|\vec{u}|^2 = (a^2 + b^2) + c^2$

$|\vec{u}|^2 = a^2 + b^2 + c^2$

$|\vec{u}| = \sqrt{a^2 + b^2 + c^2}$

Example 2: For $\vec{u} = [3, -1, 2]$…

a) sketch the position vector



b) write the vector in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$

$\vec{u} = 3\vec{i} - 1\vec{j} + 2\vec{k}$

c) find the magnitude

$|\vec{u}| = \sqrt{(3)^2 + (-1)^2 + (2)^2}$

$|\vec{u}| = \sqrt{14}$

Example 3: For the points $A(1, 3, 1)$ and $B(5, 4, -2)$

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Example 3: For the points $A(1,\ 3,\ 1)$ and $B(5,\ 4,\ -2)$

a) Find the magnitude of $\overrightarrow{AB}$

$\overrightarrow{AB} = [5-1,\ 4-3,\ -2-1]$ $\overrightarrow{AB} = [4,\ 1,\ -3]$ $\lvert\overrightarrow{AB}\rvert = \sqrt{(4)^2 + (1)^2 + (-3)^2}$ $\lvert\overrightarrow{AB}\rvert = \sqrt{26}$	**Vector between 2 points:** $\overrightarrow{P_1P_2} = [x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1]$

b) Find a unit vector, $\vec{u}$, in the same direction as $\overrightarrow{AB}$

$\vec{u} = \frac{1}{\lvert\overrightarrow{AB}\rvert}\overrightarrow{AB}$

$\vec{u} = \frac{1}{\sqrt{26}}[4,\ 1,\ -3]$

$\vec{u} = \left[\frac{4}{\sqrt{26}},\ \frac{1}{\sqrt{26}},\ \frac{-3}{\sqrt{26}}\right]$

Tools for 2-D vectors modified for 3-D vectors:

Vector Addition: $\vec{u} + \vec{v} = [u_x + v_x,\ u_y + v_y,\ u_z + v_z]$

Vector Subtraction: $\vec{u} - \vec{v} = [u_x - v_x,\ u_y - v_y,\ u_z - v_z]$

Vector between 2 points: $\overrightarrow{P_1P_2} = [x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1]$

Magnitude of a vector between 2 points: $\lvert\overrightarrow{P_1P_2}\rvert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Dot Product: for $\vec{u} = [u_1,\ u_2,\ u_3]$ and $\vec{v} = [v_1,\ v_2,\ v_3]$, $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

Example 4: Given the vectors $\vec{u} = [2,\ 3,\ -5]$, $\vec{v} = [8,\ -4,\ 3]$, and $\vec{w} = [-6,\ -2,\ 0]$, simplify each vector expression.

a) $-3\vec{v}$ $= -3[8,\ -4,\ 3]$ $= [-24,\ 12,\ -9]$	b) $\vec{u} + \vec{v} + \vec{w}$ $= [2,\ 3,\ -5] + [8,\ -4,\ 3] + [-6,\ -2,\ 0]$ $= [2 + 8 + (-6),\ 3 + (-4) + (-2),\ -5 + 3 + 0]$ $= [4,\ -3,\ -2]$
c) $\lvert\vec{u} - \vec{v}\rvert$ $\vec{u} - \vec{v} = [2,\ 3,\ -5] - [8,\ -4,\ 3]$ $\vec{u} - \vec{v} = [2 - 8,\ 3 - (-4),\ -5 - 3]$ $\vec{u} - \vec{v} = [-6,\ 7,\ -8]$ $\lvert\vec{u} - \vec{v}\rvert = \sqrt{(-6)^2 + (7)^2 + (-8)^2}$ $\lvert\vec{u} - \vec{v}\rvert = \sqrt{149}$	d) $\vec{u} \cdot \vec{v}$ $= [2,\ 3,\ -5] \cdot [8,\ -4,\ 3]$ $= 2(8) + 3(-4) + (-5)(3)$ $= 16 - 12 - 15$ $= -11$

Example 5: Determine if the vectors $\vec{a} = [6,\ 2,\ 4]$ and $\vec{b} = [9,\ 3,\ 6]$ are collinear.

Check if $\vec{b}$ is a scalar multiple of $\vec{a}$

$[6,\ 2,\ 4] = k[9,\ 3,\ 6]$

x :	y :	z :
$6 = 9k$ $k = \frac{2}{3}$	$2 = 3k$ $k = \frac{2}{3}$	$4 = 6k$ $k = \frac{2}{3}$

Therefore, $\vec{a}$ and $\vec{b}$ are collinear.

Example 6: Find a such that $[1,\ 2,\ 3]$ and $[2,\ a,\ 6]$ are collinear.

$[1,\ 2,\ 3] = k[2,\ a,\ 6]$

x :	y :	z :
$1 = 2k$ $k = \frac{1}{2}$	$2 = ak$ $2 = a\left(\frac{1}{2}\right)$ $a = 4$	$3 = 6k$ $k = \frac{1}{2}$

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Example 7: Calculate the angle between $\vec{u} = [0, -1, -4]$ and $\vec{v} = [6, 1, -2]$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

$$\cos\theta = \frac{0(6) + (-1)(1) + (-4)(-2)}{\left[\sqrt{(0)^2 + (-1)^2 + (-4)^2}\right]\left[\sqrt{(6)^2 + (1)^2 + (-2)^2}\right]}$$

$$\cos\theta = \frac{7}{(\sqrt{17})(\sqrt{41})}$$

$\theta \cong 74.6°$

Angle between 2 vectors can be found using:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Example 8: Find a vector that is orthogonal to $[3, 4, 5]$

If $[x, y, z]$ is orthogonal to $[3, 4, 5]$,

$[x, y, z] \cdot [3, 4, 5] = 0$

$3x + 4y + 5z = 0$

An infinite number of vectors would satisfy this equation. Select values for any two variables and then solve for the third.

$3(2) + 4(1) + 5z = 0$

$5z = -10$

$z = -2$

$[2, 1, -2]$ is orthogonal to $[3, 4, 5]$

Shortcut:

Make one variable equal 0, then swap the other two and change one sign.

Examples: $[0, 5, -4]$, $[0, -5, 4]$, $[5, 0, -3]$, $[-5, 0, 3]$, $[-4, 3, 0]$, and $[4, -3, 0]$ would all be orthogonal to $[3, 4, 5]$

Example 9: Find the magnitude of the projection of $\vec{u} = [3, -3, 2]$ onto $\vec{v} = [5, 2, 0]$

$$|proj_{\vec{v}}\,\vec{u}| = \left|\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}\right|$$

$$|proj_{\vec{v}}\,\vec{u}| = \left|\frac{3(5) + (-3)(2) + 2(0)}{\sqrt{5^2 + 2^2 + 0^2}}\right|$$

$$|proj_{\vec{v}}\,\vec{u}| = \left|\frac{9}{\sqrt{29}}\right|$$

$$|proj_{\vec{v}}\,\vec{u}| = \frac{9}{\sqrt{29}} \cong 1.67$$

Formula reminder:

$$|proj_{\vec{b}}\,\vec{a}| = \left|\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right|$$

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