5.3 Applications of the Dot Product Lesson with solutions

This document was composed and formatted by Annie Bernatchez.

The course material originated from www.jensenmath.ca

Warm-Up

Example 1: A desk is pushed with a force of 50 N at an angle of 45 degrees below the horizontal. If the desk is pushed 5 meters, how much work is done?

$W = \vec{f} \cdot \vec{s}$ $W = \lvert\vec{f}\rVert\vec{s}\rvert\cos\theta$ $W = 50(5)\cos 45$ $W = 250\left(\frac{1}{\sqrt{2}}\right)$ $W = \frac{250}{\sqrt{2}}$ or $125\sqrt{2}$ joules	**Remember:** Mechanical work is the product of the magnitude of the displacement travelled by an object and the magnitude of the force applied in the direction of the motion.

Part 1: Angle Between 2 Vectors

To determine the angle between two vectors, you can rearrange the dot product formula, $\vec{a} \cdot \vec{b} = \lvert\vec{a}\rVert\vec{b}\rvert\cos\theta$, to isolate $\cos\theta$:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\lvert\vec{a}\rVert\vec{b}\rvert}$$

Example 2: Determine the angle between each pair of vectors.

a) $\vec{g} = [5, 1]$ and $\vec{h} = [-3, 8]$	b) $\vec{a} = [-3, 6]$ and $\vec{b} = [4, 2]$
$\cos\theta = \frac{\vec{g} \cdot \vec{h}}{\lvert\vec{g}\rVert\vec{h}\rvert}$ $\cos\theta = \frac{5(-3) + 1(8)}{\left(\sqrt{(5)^2 + (1)^2}\right)\left(\sqrt{(-3)^2 + (8)^2}\right)}$ $\cos\theta = \frac{-7}{(\sqrt{26})(\sqrt{73})}$ $\theta \cong 99.2°$	$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\lvert\vec{a}\rVert\vec{b}\rvert}$ $\cos\theta = \frac{(-3)(4) + 6(2)}{\left(\sqrt{(-3)^2 + (6)^2}\right)\left(\sqrt{(4)^2 + (2)^2}\right)}$ $\cos\theta = \frac{0}{(\sqrt{45})(\sqrt{20})}$ $\theta = 90°$

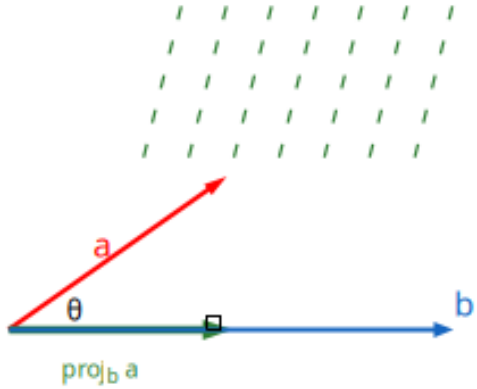
Part 2: Vector Projections

You can think of a vector projection like a shadow. The vertical arrows in the diagrams represent light from above.

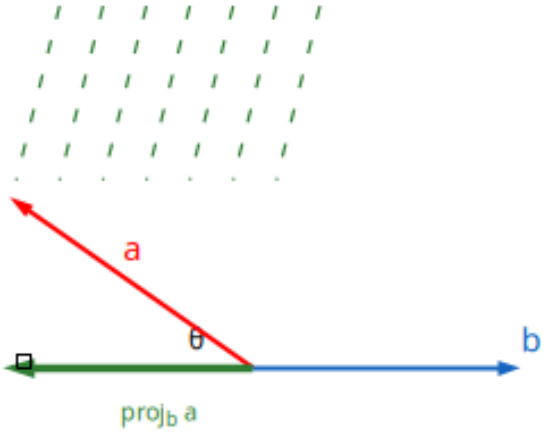
Think of the projection of $\vec{a}$ on $\vec{b}$ as the shadow that $\vec{a}$ casts on $\vec{b}$.

If the angle between $\vec{a}$ and $\vec{b}$ is less than 90°, then the projection of $\vec{a}$ on $\vec{b}$, or $proj_{\vec{b}}\vec{a}$, is the vector component of $\vec{a}$ in the direction of $\vec{b}$.

Copyright © 2026 Annie Bernatchez—All rights reserved.

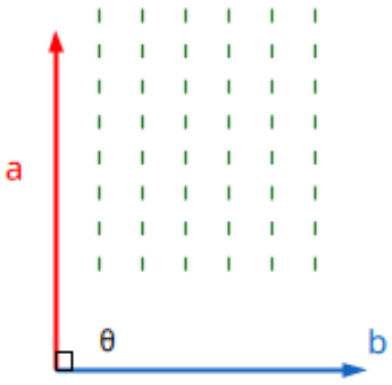


If the angle between $\vec{a}$ and $\vec{b}$ is between 90° and 180°, the direction of $proj_{\vec{b}}\,\vec{a}$ is in the opposite direction of $\vec{b}$.



If $\vec{a}$ is perpendicular to $\vec{b}$, then $\vec{a}$ casts 'no shadow' on to $\vec{b}$. So if $\theta = 90°$, $proj_{\vec{b}}\,\vec{a} = 0$.

Note: This is why the dot product $\vec{a} \cdot \vec{b}$ would be zero for perpendicular vectors.



Formulas for Vector Projection:

Geometric Formulas: $proj_{\vec{b}}\,\vec{a} = |\vec{a}|\cos\theta\,(\hat{b})$ OR $proj_{\vec{b}}\,\vec{a} = |\vec{a}|\cos\theta\left(\frac{\vec{b}}{|\vec{b}|}\right)$

Cartesian Formulas: $proj_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}(\hat{b})$ OR $proj_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\left(\frac{\vec{b}}{|\vec{b}|}\right)$ OR $proj_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}(\vec{b})$

Copyright © 2026 Annie Bernatchez—All rights reserved.

Formulas for Magnitude of Vector Projection:

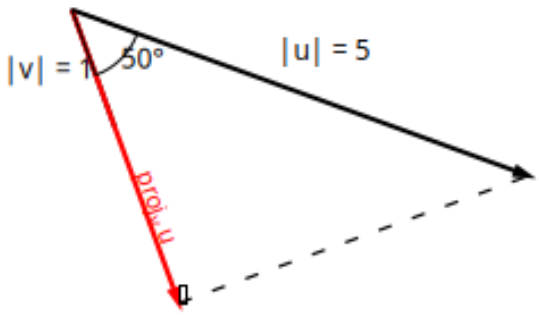
If $0° < \theta < 90°$: $|proj_{\vec{b}}\ \vec{a}| = |\vec{a}|\cos\theta$ OR $|proj_{\vec{b}}\ \vec{a}| = \left|\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right|$

If $90° < \theta < 180°$: $|proj_{\vec{b}}\ \vec{a}| = -|\vec{a}|\cos\theta$

Note: $\frac{\vec{b}}{|\vec{b}|}$ *is a unit vector in the direction of* $\vec{b}$. *Sometimes the symbol* $\hat{b}$ *is used to denote a unit vector in the direction of* $\vec{b}$.

Example 3: Determine the following projections of one vector on another.

a) Determine the projection of $\vec{u}$ on $\vec{v}$



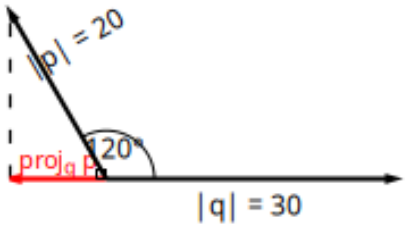
$proj_{\vec{v}}\ \vec{u} = |\vec{u}|\cos\theta(\hat{v})$

$proj_{\vec{v}}\ \vec{u} = 5\cos 50(\hat{v})$

$proj_{\vec{v}}\ \vec{u} \cong 3.2\hat{v}$

3.2 units in the same direction as $\vec{v}$

b) Determine $proj_{\vec{q}}\ \vec{p}$



$proj_{\vec{q}}\ \vec{p} = |\vec{p}|\cos\theta(\hat{q})$

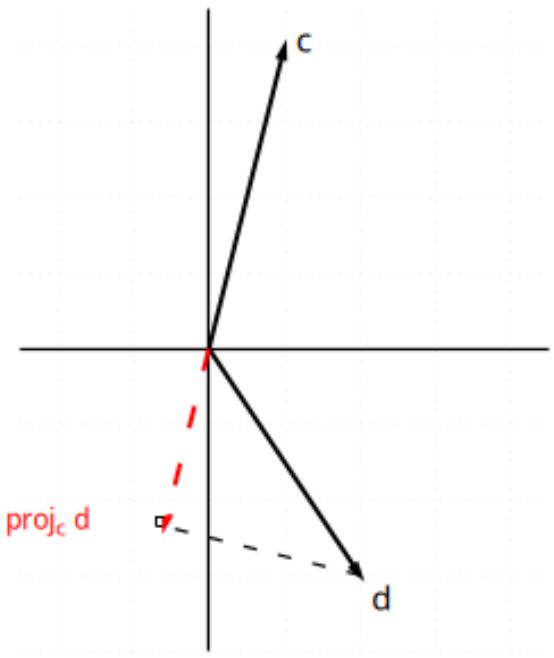
$proj_{\vec{q}}\ \vec{p} = 20\cos 120(\hat{q})$

$proj_{\vec{q}}\ \vec{p} \cong -10\hat{q}$

10 units in the opposite direction as $\vec{q}$

Copyright © 2026 Annie Bernatchez—All rights reserved.

c) Determine the projection of $\vec{d} = [2, -3]$ on $\vec{c} = [1, 4]$

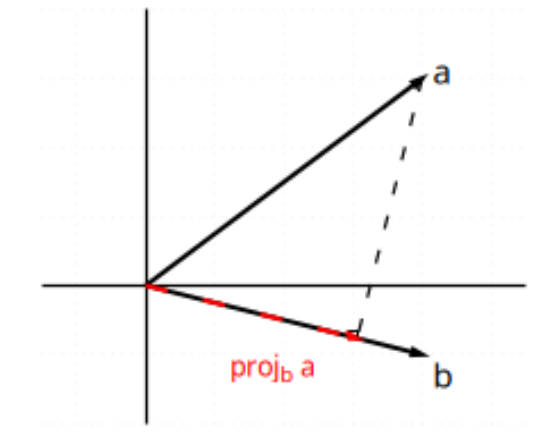


$$proj_{\vec{c}}\,\vec{d} = \frac{\vec{d}\cdot\vec{c}}{\vec{c}\cdot\vec{c}}(\vec{c})$$
$$proj_{\vec{c}}\,\vec{d} = \left(\frac{2(1)+(-3)(4)}{1^2+4^2}\right)[1, 4]$$
$$proj_{\vec{c}}\,\vec{d} = \left(\frac{-10}{17}\right)[1, 4]$$
$$proj_{\vec{c}}\,\vec{d} = \left[\frac{-10}{17}, \frac{-40}{17}\right]$$

d) Find the magnitude of the projection of $\vec{a} = [4, 3]$ on $\vec{b} = [4, -1]$

$$|proj_{\vec{b}}\,\vec{a}| = \left|\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right|$$
$$|proj_{\vec{b}}\,\vec{a}| = \left|\frac{4(4)+(3)(-1)}{\sqrt{(4)^2+(-1)^2}}\right|$$
$$|proj_{\vec{b}}\,\vec{a}| = \frac{13}{\sqrt{17}}$$

e) Find the projection of $\vec{a} = [4, 3]$ on $\vec{b} = [4, -1]$



$$proj_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}(\vec{b})$$
$$proj_{\vec{b}}\,\vec{a} = \frac{4(4)+(3)(-1)}{(4)^2+(-1)^2}[4, -1]$$
$$proj_{\vec{b}}\,\vec{a} = \frac{13}{17}[4, -1]$$
$$proj_{\vec{b}}\,\vec{a} = \left[\frac{52}{17}, \frac{-13}{17}\right]$$

Part 3: Dot Product with Sales

Example 4: A shoe store sold 350 pairs of Nike shoes and 275 pairs of Adidas shoes in a year. Nike shoes sell for

Copyright © 2026 Annie Bernatchez—All rights reserved.

$175 and Adidas shoes sell for $250.

a) Write a Cartesian vector, $\vec{s}$, to represent the numbers of pairs of shoes sold.

$\vec{s} = [350,\ 275]$

b) Write a Cartesian vector, $\vec{p}$, to represent the prices of the shoes.

$\vec{p} = [175,\ 250]$

c) Find the dot product $\vec{s} \cdot \vec{p}$. What does this dot product represent?

$\vec{s} \cdot \vec{p} = 350(175) + 275(250)$

$\vec{s} \cdot \vec{p} = 130\,000$

The dot product represents the revenue, $130 000, from sales of the shoes.

Copyright © 2026 Annie Bernatchez—All rights reserved.