5.1 Cartesian Vectors Lesson with solutions

This document was composed and formatted by Annie Bernatchez.

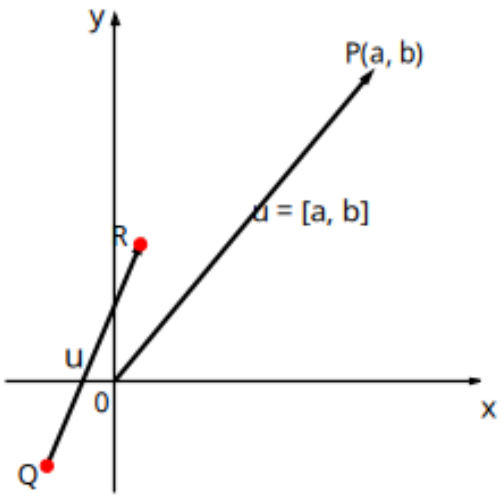
The course material originated from www.jensenmath.ca

Mathematicians started using coordinates to analyze physical situations in about the fourteenth century. However, a great deal of the credit for developing the methods used with coordinate systems should be given to the French mathematician Rene Descartes (1596-1650). Descartes was the first to realize that using a coordinate system would allow for the use of algebra in geometry. Since then, this idea has become important in the development of mathematical ideas in many areas. For our purposes, using algebra in this way leads us to the consideration of ideas involving vectors that otherwise would not be possible.

Part 1: What are Cartesian (algebraic) Vectors?

Suppose $\vec{u}$ is any vector with endpoints Q and R. We identify $\overrightarrow{QR}$ as a Cartesian vector because its endpoints can be defined using Cartesian coordinates.

If we translate $\vec{u}$ so that its tail is at the origin, O, then its head will be at some point $P(a, b)$. Then we define this Cartesian vector as position vector $[a, b]$.



Note: Use square brackets to distinguish between a point (a, b) and a position vector $[a, b]$

Part 2: Resolving Cartesian Vectors into Unit Vectors

A second way of writing $\overrightarrow{OP} = [a, b]$ is with the use of unit vectors $\vec{i}$ and $\vec{j}$.

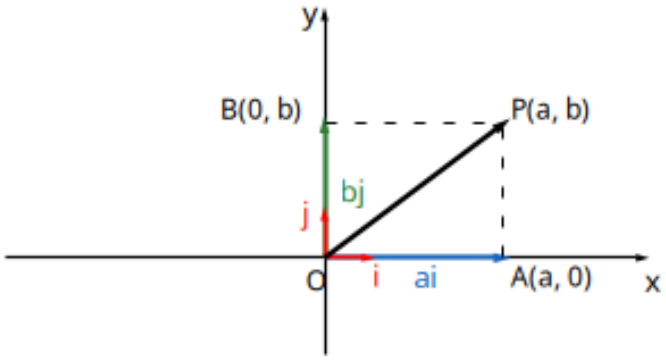
$\vec{i} = [1, 0]$ and $\vec{j} = [0, 1]$

Both of these vectors have magnitude of 1 and lie along the positive x and y axes respectively.

In the diagram, $\overrightarrow{OA} = [a, 0] = a\vec{i}$

$\overrightarrow{OB} = [0, b] = b\vec{j}$

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Therefore, using the triangle law of addition, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} = a\vec{i} + b\vec{j}$

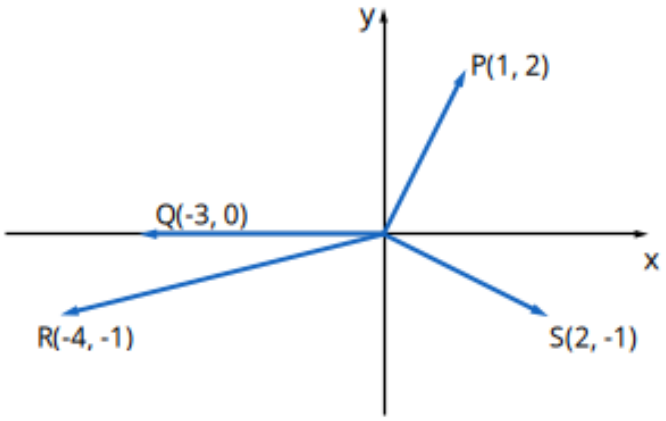
It follows that $[a, b] = a\vec{i} + b\vec{j}$

Representations of Vectors in $\mathbb{R}^2$

The position vector $\overrightarrow{OP}$ can be represented as either $\overrightarrow{OP} = [a, b]$ or $\overrightarrow{OP} = a\vec{i} + b\vec{j}$ where $O(0, 0)$ is the origin and $P(a, b)$ is any point on the plane.

Example 1: Practice representing vectors in two equivalent forms.

a) Four position vectors, $\overrightarrow{OP} = [1, 2]$, $\overrightarrow{OQ} = [-3, 0]$, $\overrightarrow{OR} = [-4, -1]$, and $\overrightarrow{OS} = [2, -1]$ are shown. Write each of these vectors using the unit vectors $\vec{i}$ and $\vec{j}$.



$\overrightarrow{OP} = \vec{i} + 2\vec{j}$ $\overrightarrow{OQ} = -3\vec{i}$

$\overrightarrow{OR} = -4\vec{i} - \vec{j}$ $\overrightarrow{OS} = 2\vec{i} - \vec{j}$

b) The vectors $\overrightarrow{OA} = -\vec{i}$, $\overrightarrow{OB} = \vec{i} + 5\vec{j}$, and $\overrightarrow{OC} = -5\vec{i} + 2\vec{j}$ have been written using the unit vectors $\vec{i}$ and $\vec{j}$. Write them in component form $[a, b]$.

$\overrightarrow{OA} = [-1, 0]$ $\overrightarrow{OB} = [1, 5]$ $\overrightarrow{OC} = [-5, 2]$

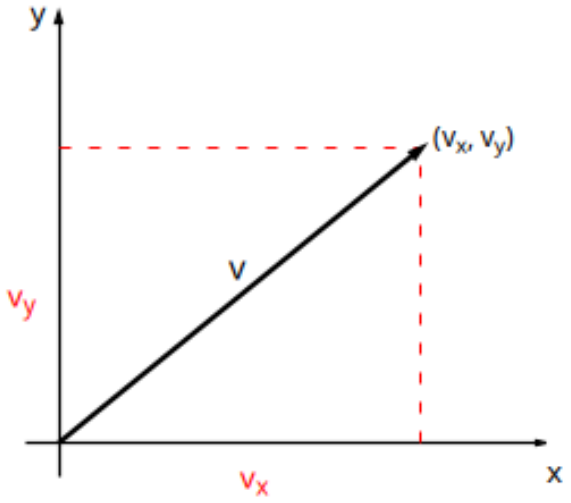
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Part 3: Magnitude of Vectors

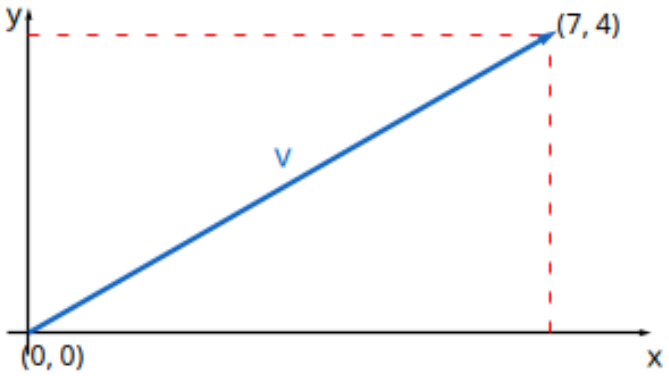
Any Cartesian vector $\vec{v} = [v_x, v_y]$ can be translated so its tail is at the origin, $(0, 0)$, and its head is at the point (v_x, v_y). To find the magnitude of the vector, use the formula for the distance between two points:

$$|\vec{v}| = \sqrt{(v_x - 0)^2 + (v_y - 0)^2}$$

$$|\vec{v}| = \sqrt{(v_x)^2 + (v_y)^2}$$



Example 2: Find the magnitude of vector $\vec{v} = [7, 4]$



$|\vec{v}| = \sqrt{(7)^2 + (4)^2}$

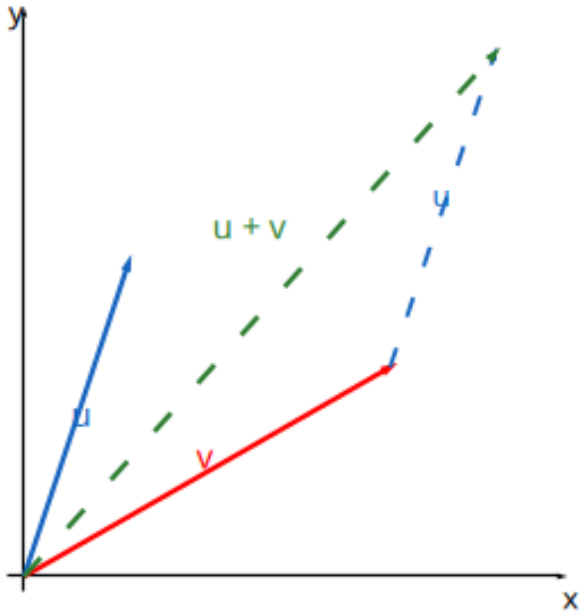
$|\vec{v}| = \sqrt{65}$

Part 4: Adding and Subtracting Vectors

Rule: $\vec{v} + \vec{u} = [v_x + u_x, v_y + u_y]$

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Example 3: If $\vec{v} = [7, 4]$ and $\vec{u} = [2, 6]$. Find $\vec{v} + \vec{u}$.

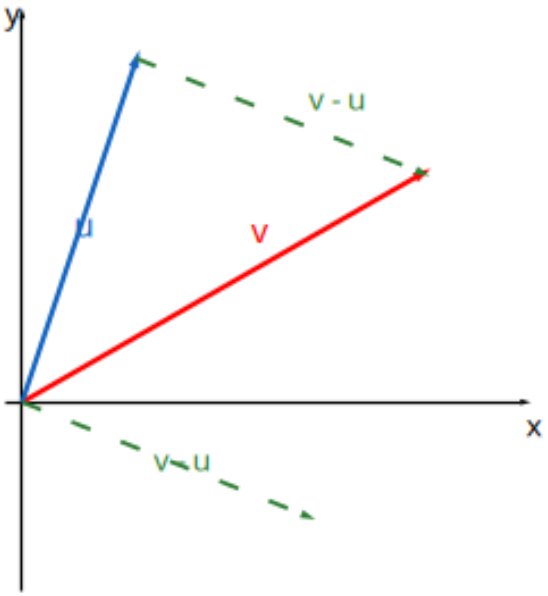


$\vec{v} + \vec{u} = [7 + 2, 4 + 6]$

$\vec{v} + \vec{u} = [9, 10]$

Rule: $\vec{v} - \vec{u} = [v_x - u_x, v_y - u_y]$

Example 4: If $\vec{v} = [7, 4]$ and $\vec{u} = [2, 6]$. Find $\vec{v} - \vec{u}$.



$\vec{v} - \vec{u} = [7 - 2, 4 - 6]$

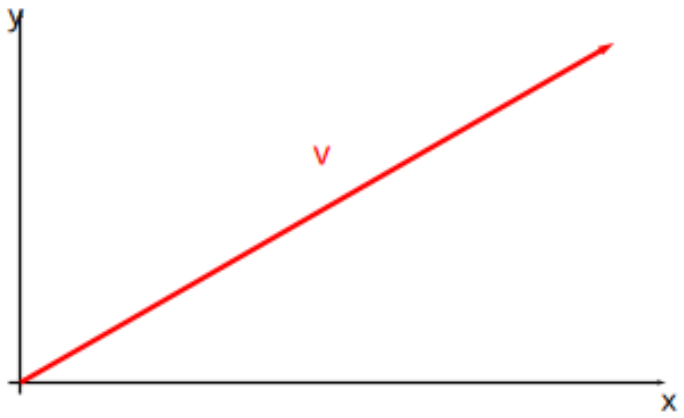
$\vec{v} - \vec{u} = [5, -2]$

Part 5: Multiplying a Vector by a Scalar

For $\vec{v} = [v_x, v_y]$, $k\vec{v} = [kv_x, kv_y]$

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Example 5: If $\vec{v} = [7, 4]$, find $2\vec{v}$.



$2\vec{v} = [2(7), 2(4)]$

$2\vec{v} = [14, 8]$

Recall: Two vectors that are **collinear** can be drawn so they lie on the same line. In other words, they are scalar multiples of one another.

If $\vec{u} = k\vec{v}$ then $\frac{u_x}{v_x} = \frac{u_y}{v_y}$

Example 6: Are $\vec{u} = \left[3, \frac{5}{6}\right]$ and $\vec{v} = \left[\frac{-3}{2}, \frac{-5}{12}\right]$ collinear?

$\frac{u_x}{v_x} = \frac{u_y}{v_y}$

$\frac{3}{\left(\frac{-3}{2}\right)} = \frac{\left(\frac{5}{6}\right)}{\left(\frac{-5}{12}\right)}$

$\frac{6}{-3} = \frac{60}{-30}$

$-2 = -2$

$\vec{u} = -2\vec{v}$, therefore $\vec{u}$ and $\vec{v}$ are collinear.

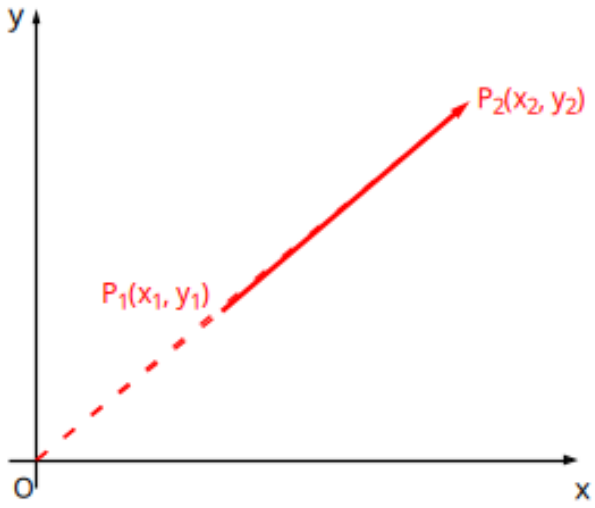
Part 6: Cartesian Vector Between 2 Points

Notice that $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$

Therefore, $\overrightarrow{P_1P_2} = [x_2 - x_1, y_2 - y_1]$

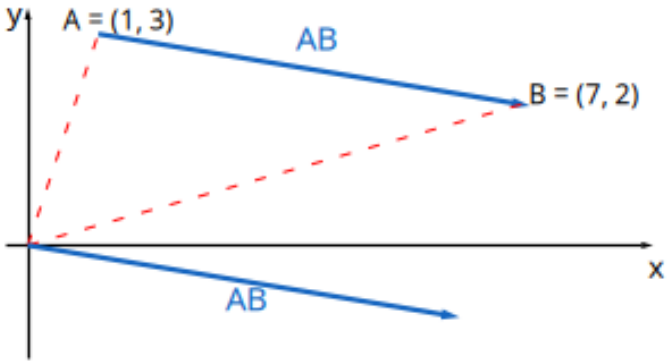
Or, $\overrightarrow{P_1P_2} = [x_{tip} - x_{tail}, y_{tip} - y_{tail}]$

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Example 7: Find the coordinates and the magnitude of each vector.

a) $\overrightarrow{AB}$, for $A(1,\ 3)$ and $B(7,\ 2)$



$\overrightarrow{AB} = [7 - 1,\ 2 - 3]$

$\overrightarrow{AB} = [6,\ -1]$

$|\overrightarrow{AB}| = \sqrt{(6)^2 + (-1)^2}$

$|\overrightarrow{AB}| = \sqrt{37}$

b) $\overrightarrow{CD}$, for $C(-10,\ 0)$ and $D(0,\ 10)$

$\overrightarrow{CD} = [0 - (-10),\ 10 - 0]$

$\overrightarrow{CD} = [10,\ 10]$

$|\overrightarrow{CD}| = \sqrt{(10)^2 + (10)^2}$

$|\overrightarrow{CD}| = \sqrt{200}$

$|\overrightarrow{CD}| = 10\sqrt{2}$

c) $\overrightarrow{EF}$, for $E(4,\ -3)$ and $F(1,\ -7)$

$\overrightarrow{EF} = [1 - 4,\ -7 - (-3)]$

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$\overrightarrow{EF} = [-3, \ -4]$

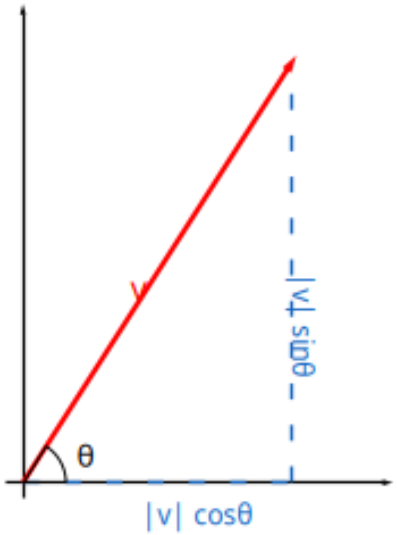
$|\overrightarrow{EF}| = \sqrt{(-3)^2 + (-4)^2}$

$|\overrightarrow{EF}| = \sqrt{25}$

$|\overrightarrow{EF}| = 5$

Part 7: Geometric Vectors in Cartesian Form

To write a geometric vector $\vec{v}$ in Cartesian form, you need to use trigonometry. In the previous unit, we found that the magnitude of the horizontal component is $|\vec{v}|\cos\theta$, and the magnitude of the vertical component is $|\vec{v}|\sin\theta$, where θ is the angle $\vec{v}$ makes with the horizontal, or the positive x-axis.



Example 8: Write a force of 200 N at 20° to the horizontal in Cartesian form.

$\vec{F} = [|\vec{F}|\cos\theta, \ |\vec{F}|\sin\theta]$

$\vec{F} = [200\cos 20, \ 200\sin 20]$

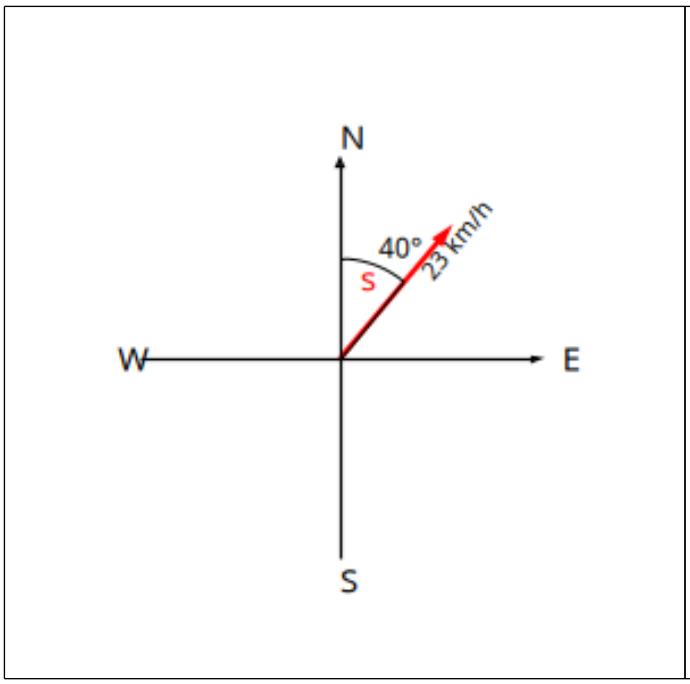
$\vec{F} \cong [187.9, \ 68.4]$

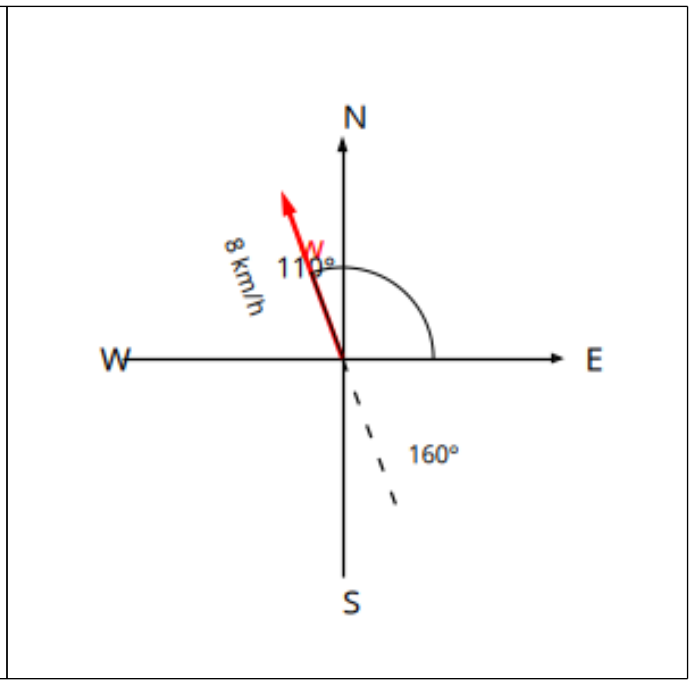
Example 9: A ship's course is set to travel at 23 km/h, relative to the water, on a heading of 040°. A current of 8 km /h is flowing from a bearing of 160°.

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Example 9: A ship's course is set to travel at 23 km/h, relative to the water, on a heading of 040°. A current of 8 km /h is flowing from a bearing of 160°.

a) Write each vector as a Cartesian (algebraic) vector.





$\vec{s} = [23\cos 50,\ 23\sin 50]$

$\vec{w} = [8\cos 110,\ 8\sin 110]$

b) Determine the resultant velocity of the ship.

$\vec{r} = \vec{s} + \vec{w}$

$\vec{r} = [23\cos 50 + 8\cos 110,\ \ 23\sin 50 + 8\sin 110]$

$\vec{r} \cong [12.048,\ 25.137]$

Magnitude:

$|\vec{r}| \cong \sqrt{(12.048)^2 + (25.137)^2}$

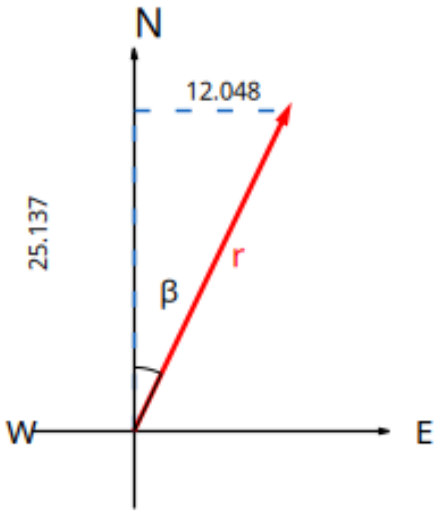
$|\vec{r}| \cong 27.875$

Direction:

$\beta = \tan^{-1}\left(\frac{12.048}{25.137}\right)$

$\beta \cong 25.6°$

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The ship is travelling at a velocity of about 27.875 km/h at a bearing of 025.6°.

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