4.3 Multiplication of a Vector by a Scalar Lesson with solutions

This document was composed and formatted by Annie Bernatchez.

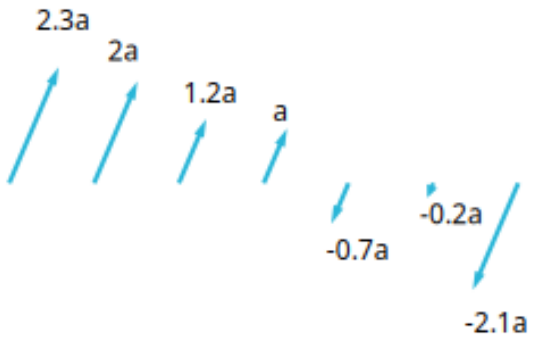
The course material originated from www.jensenmath.ca

Part 1: Scalar Multiplication

In this section we will multiply a vector, $\vec{a}$, by a number k to produce a new vector $k\vec{a}$. The number k used for multiplication is called a SCALAR and can be any real number.

Multiplying a vector by different values of a scalar k can affect the **magnitude** AND **direction** of the vector.

When two vectors are parallel, these vectors are described as being colinear. They are described this way because they can be translated to be on the same straight line. Scalar multiples of vectors are colinear.



Rules:

1) If $k > 0$, then $k\vec{a}$ is in the same direction as $\vec{a}$ with magnitude $k|\vec{a}|$

a) If $0 < k < 1$, $\vec{a}$ is shortened $\rightarrow \left|\frac{1}{2}\vec{a}\right| = \frac{1}{2}|\vec{a}|$

b) If $k > 1$, $\vec{a}$ is lengthened $\rightarrow |2\vec{a}| = 2|\vec{a}|$

2) If $k < 0$, then $k\vec{a}$ is in the opposite direction as $\vec{a}$ with magnitude $|k||\vec{a}|$

a) If $-1 < k < 0$, $\vec{a}$ is shortened and changes to the opposite direction $\rightarrow \left|-\frac{1}{2}\vec{a}\right| = \frac{1}{2}|\vec{a}|$

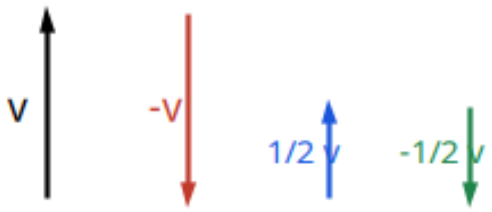
b) If $k < -1$, $\vec{a}$ is lengthened and changes to the opposite direction $\rightarrow |-2\vec{a}| = 2|\vec{a}|$

3) If $k = 0$, the result is the zero vector $\vec{0}$

4) The distributive property for vector multiplication is $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

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Example 1: An airplane is heading due north at 1000 km/h. The airplane's velocity is represented by $\vec{v}$. Draw the vectors $-\vec{v}$, $\frac{1}{2}\vec{v}$, and $-\frac{1}{2}\vec{v}$ and give an interpretation for each.

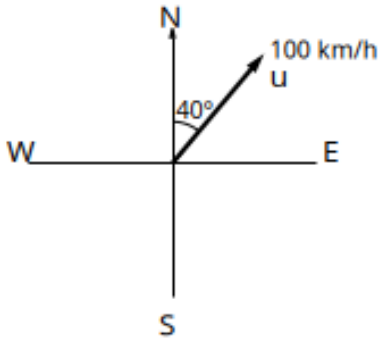


$-\vec{v}$; velocity vector heading due south at 1000 km/h

$\frac{1}{2}\vec{v}$; velocity vector heading due north at 500 km/h

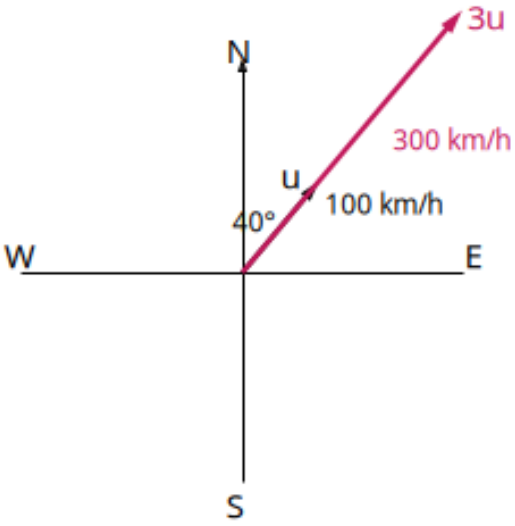
$-\frac{1}{2}\vec{v}$; velocity vector heading due south at 500 km/h

Example 2: Consider vector $\vec{u}$ with magnitude $|\vec{u}| = 100$ km/h, at a quadrant bearing of $N40°E$. Draw a vector with each scalar multiplication. Describe the resulting vector.



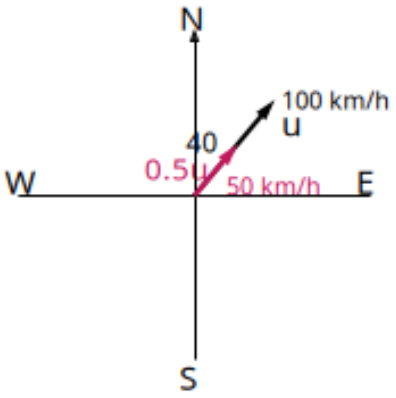
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a) $3\vec{u}$



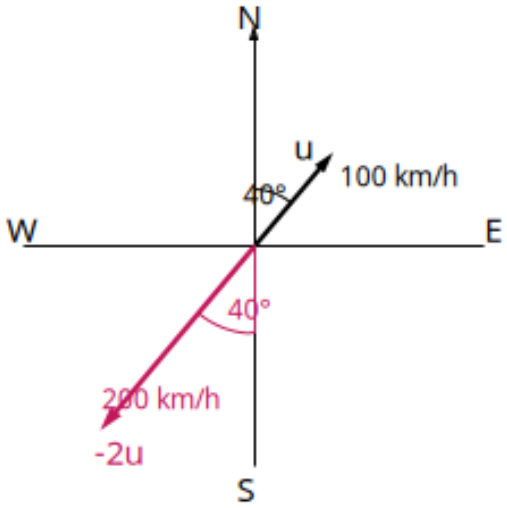
The velocity is 300 km/h at a quadrant bearing of $N40°E$.

b) $0.5\vec{u}$



The velocity is 50 km/h at a quadrant bearing of $N40°E$.

c) $-2\vec{u}$



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The velocity is 200 km/h at a quadrant bearing of $S40°W$.

Vector Properties for Scalar Multiplication

Distributive Property: For any scalar $k \in \mathbb{R}$ and any vectors $\vec{u}$ and $\vec{v}$, $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

Associative Property: For any scalars $a, b \in \mathbb{R}$ and any vector $\vec{v}$, $(ab)\vec{v} = a(b\vec{v})$

Identity Property: For any vector $\vec{v}$, $1\vec{v} = \vec{v}$

Example 3: If $\vec{x} = 3\vec{i} - 4\vec{j} + \vec{k}$, $\vec{y} = \vec{j} - 5\vec{k}$, and $\vec{z} = -\vec{i} - \vec{j} + 4\vec{k}$, determine each of the following in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

a) $\vec{x} + \vec{y}$

$$= (3\vec{i} - 4\vec{j} + \vec{k}) + (\vec{j} - 5\vec{k})$$
$$= 3\vec{i} - 4\vec{j} + \vec{k} + \vec{j} - 5\vec{k}$$
$$= 3\vec{i} - 3\vec{j} - 4\vec{k}$$

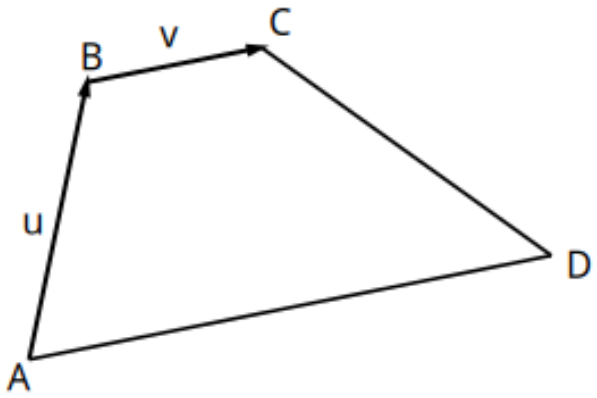
b) $\vec{x} - \vec{y}$

$$= (3\vec{i} - 4\vec{j} + \vec{k}) - (\vec{j} - 5\vec{k})$$
$$= 3\vec{i} - 4\vec{j} + \vec{k} - \vec{j} + 5\vec{k}$$
$$= 3\vec{i} - 5\vec{j} + 6\vec{k}$$

c) $\vec{x} - 2\vec{y} + 3\vec{z}$

$$= (3\vec{i} - 4\vec{j} + \vec{k}) - 2(\vec{j} - 5\vec{k}) + 3(-\vec{i} - \vec{j} + 4\vec{k})$$
$$= 3\vec{i} - 4\vec{j} + \vec{k} - 2\vec{j} + 10\vec{k} - 3\vec{i} - 3\vec{j} + 12\vec{k}$$
$$= -9\vec{j} + 23\vec{k}$$

Example 4: In trapezoid ABCD, $BC \parallel AD$ and $AD = 3BC$. Let $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{BC} = \vec{v}$. Express $\overrightarrow{AD}$, $\overrightarrow{BD}$, and $\overrightarrow{CD}$ as linear combinations of $\vec{u}$ and $\vec{v}$.



$\overrightarrow{AD} = 3\vec{v}$

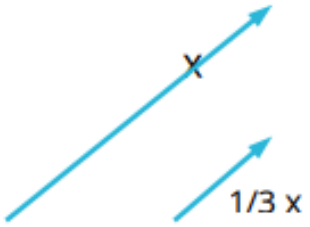
$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -\vec{u} + 3\vec{v}$

$\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AD} = -\vec{v} + (-\vec{u}) + 3\vec{v} = 2\vec{v} - \vec{u}$

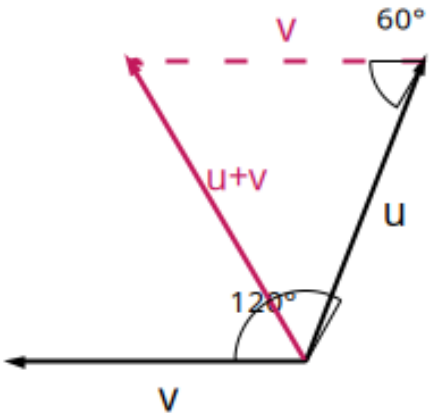
Part 2: Creating Unit Vectors

It is sometimes useful to multiply the nonzero vector $\vec{x}$ by the scalar $\frac{1}{|\vec{x}|}$. When we multiply $\vec{x}$ by $\frac{1}{|\vec{x}|}$, we get the vector $\frac{1}{|\vec{x}|}\vec{x}$. This is a vector of length one and is called a unit vector, which points in the same direction as $\vec{x}$. The concept of unit vectors will prove to be very useful when we discuss applications of vectors.

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Example 5: Given that $|\vec{u}| = 4$ and $|\vec{v}| = 5$ and the angle between $\vec{u}$ and $\vec{v}$ is 120°, determine the unit vector in the same direction as $\vec{u} + \vec{v}$.



$$|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(60)$$
$$|\vec{u} + \vec{v}|^2 = (4)^2 + (5)^2 - 2(4)(5)\cos(60)$$
$$|\vec{u} + \vec{v}|^2 = 41 - 40\left(\frac{1}{2}\right)$$
$$|\vec{u} + \vec{v}|^2 = 21$$
$$|\vec{u} + \vec{v}| = \sqrt{21}$$

To create a unit vector in the same direction as $\vec{u} + \vec{v}$, multiply by the scalar equal to $\frac{1}{|\vec{u} + \vec{v}|}$

Therefore, the unit vector is $\frac{1}{\sqrt{21}}(\vec{u} + \vec{v}) = \frac{1}{\sqrt{21}}\vec{u} + \frac{1}{\sqrt{21}}\vec{v}$

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