2.6 Optimization Problems Lesson with solutions

This document was composed and formatted by Annie Bernatchez.

The course material originated from www.jensenmath.ca

Tips for Optimization Problems:

- Diagrams can be helpful
- Identify the independent variable and express all other variables in terms of it
- Define a function in terms of the independent variable
- Identify any restriction on the variable
- Solve for $f'(x) = 0$ to identify critical points
- Check critical points and endpoints

Optimization Warm Up:

A lifeguard has 200 meters of rope and some buoys with which she intends to enclose a rectangular area at a lake for swimming. The beach will form one side of the rectangle, with the rope forming the other 3 sides. Find the dimensions that will produce the maximum enclosed area.

$$\text{width} = x$$
$$\text{length} = 200 - 2x$$
$$A = (\text{length})(\text{width})$$
$$A = x(200 - 2x)$$
$$A = 200x - 2x^2$$

Note: The domain of this function is restricted to values $0 < x < 100$ because there is only 200 meters of rope to use.

To determine the max area, test endpoints of the interval as well as any critical numbers.

Critical Number(s):

$$A'(x) = 200 - 4x$$
$$0 = 200 - 4x$$
$$4x = 200$$
$$x = 50 \text{ is a critical number}$$

Tests:

$$A(0) = 0\,[200 - 2(0)]$$
$$A(0) = 0 \text{ m}^2$$
$$A(50) = 50\,[200 - 2(50)]$$
$$A(50) = 5000 \text{ m}^2$$
$$A(100) = 100\,[200 - 2(100)]$$
$$A(100) = 0 \text{ m}^2$$

Therefore, the max area of 5000 m^2 occurs when the width is 50m and the length is 100m.

Example 1: A cardboard box with a square base is to have a volume of 8 Liters (1 L = 1000 cm^3). Find the dimensions that will minimize the amount of cardboard to be used. What is the minimum surface area?

$$SA = 2x^2 + 4xh$$

From Volume Equation:

$$V = (\text{area of base})(\text{height})$$
$$8000 = x^2 h$$
$$h = \frac{8000}{x^2}$$

Express h in terms of x using the volume equation

$$SA = 2x^2 + 4x\left(\frac{8000}{x^2}\right)$$
$$SA = 2x^2 + 32000x^{-1}$$

Find min value by finding zero(s) of the first derivative:

$$SA'(x) = 4x - 32000x^{-2}$$
$$0 = 4x - 32000x^{-2}$$
$$32000x^{-2} = 4x$$

Copyright © 2026 Annie Bernatchez—All rights reserved.

$$32000 = 4x^3$$
$$8000 = x^3$$
$$x = 20 \text{ cm}$$

Verify it is a min using the second derivative test:

$$SA''(x) = 4 + 64000x^{-3}$$
$$SA''(20) = 4 + 64000(20)^{-3}$$
$SA''(20) = 12$, therefore SA is concave up at $x = 20$, and there is a local min point.

Solve for h when $x = 20$:

$$h = \frac{8000}{(20)^2}$$
$$h = 20$$

Find the minimum surface area:

$$SA(20) = 2(20)^2 + 32000(20)^{-1}$$
$$SA(20) = 2400 \text{ cm}^2$$

Therefore, a minimum surface area of 2400 cm^2 can be obtained when the dimensions of the box are 20 by 20 by 20 cm.

Example 2: A soup can of volume 500 cm^3 is to be constructed. The material for the top costs 0.4¢/cm^2 while the material for the bottom and sides costs 0.2¢/cm^2. Find the dimensions that will minimize the cost of producing the can. What is the min cost?

$$SA = 2\pi r^2 + 2\pi rh$$
$$SA = \pi r^2 + \pi r^2 + 2\pi rh$$

From Volume Equation:

$$V = \pi r^2 h$$
$$500 = \pi r^2 h$$
$$h = \frac{500}{\pi r^2}$$
$$SA(r) = \pi r^2 + \pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)$$
$$SA(r) = \pi r^2 + \pi r^2 + \frac{1000}{r}$$
$$C(r) = 0.4(\pi r^2) + 0.2(\pi r^2) + 0.2\left(\frac{1000}{r}\right)$$
$$C(r) = 0.6\pi r^2 + \frac{200}{r}$$
$$C'(r) = 1.2\pi r - \frac{200}{r^2}$$
$$0 = 1.2\pi r - \frac{200}{r^2}$$
$$\frac{200}{r^2} = 1.2\pi r$$
$$\frac{200}{1.2\pi} = r^3$$
$$r = \sqrt[3]{\frac{200}{1.2\pi}}$$
$$r = 3.76 \text{ cm}$$

2nd Derivative Test

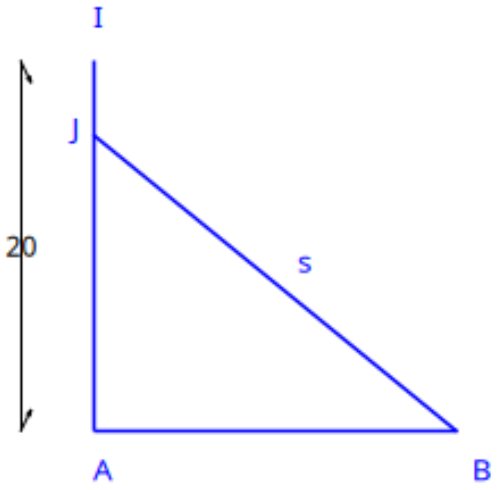
$$C''(r) = 1.2\pi + \frac{400}{r^3}$$
$C''(3.76) = 11.29$; therefore at the point $(3.76,\ C(3.76))$, $C(r)$ is concave up and the point is a MIN point.
$$C(3.76) = 79.84 \text{ cents}$$

A min cost of 79.84 cents can be obtained by using dimensions $r = 3.76$ cm and $h = \frac{500}{\pi(3.76)^2} = 11.26$ cm

Example 3: Ian and Ada are both training for a marathon. Ian's house is located 20 km north of Ada's house. At 9:00 am one Saturday, Ian leaves his house and jogs south at 8 km/h. At the same time, Ada leaves her house and jogs east at 6 km/h. When are Ian and Ada closest together, given that they both run for 2.5 hours?

$$s^2 = JA^2 + AB^2$$
$$s = \sqrt{JA^2 + AB^2}$$

Copyright © 2026 Annie Bernatchez—All rights reserved.



Express s in terms of time (t)

$$JA = 20 - 8t$$
$$AB = 6t$$
$$s = \sqrt{(20 - 8t)^2 + (6t)^2}$$
$$s = \left[400 - 320t + 64t^2 + 36t^2\right]^{\frac{1}{2}}$$
$$s = \left(100t^2 - 320t + 400\right)^{\frac{1}{2}}$$

Find Critical Number(s):

$$s'(t) = \frac{1}{2}\left(100t^2 - 320t + 400\right)^{-\frac{1}{2}}(200t - 320)$$
$$s'(t) = \frac{200t - 320}{2\sqrt{100t^2 - 320t + 400}}$$
$$s'(t) = \frac{100t - 160}{\sqrt{100t^2 - 320t + 400}}$$
$$0 = \frac{100t - 160}{\sqrt{100t^2 - 320t + 400}}$$
$$0 = 100t - 160$$
$$160 = 100t$$
$$t = 1.6$$

Check endpoints of interval and critical number to determine minimum value:

$$s(0) = \sqrt{[20 - 8(0)]^2 + [6(0)]^2} = 20$$
$$s(1.6) = \sqrt{[20 - 8(1.6)]^2 + [6(1.6)]^2} = 12$$
$$s(2.5) = \sqrt{[20 - 8(2.5)]^2 + [6(2.5)]^2} = 15$$

Therefore, the minimum distance between Ada and Ian occurs after 1.6 hours (10:36 am).

Copyright © 2026 Annie Bernatchez—All rights reserved.