2.3 Factor Theorem Lesson with solutions

This document was composed and formatted by Annie Bernatchez.

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In this section, you will learn how to determine the factors of a polynomial function of degree 3 or greater.

Part 1: Remainder Theorem Refresher

a) Use the remainder theorem to determine the remainder when $f(x) = x^3 + 4x^2 + x - 6$ is divided by $x + 2$

$$\begin{aligned} f(-2) &= (-2)^3 + 4(-2)^2 + (-2) - 6 \\ &= -8 + 16 - 2 - 6 \\ &= 0 \end{aligned}$$

The remainder when divided by $x + 2$ is 0. This means that $x + 2$ is a factor of the dividend.

> **Remainder Theorem:** When a polynomial function $P(x)$ is divided by $x - b$, the remainder is $P(b)$; and when it is divided by $ax - b$, the remainder is $P\left(\frac{b}{a}\right)$, where a and b are integers, and $a \neq 0$.

b) Verify your answer to part a) by completing the division using long division or synthetic division.

Note: I chose synthetic since it is a linear divisor of the form $x - b$.

$$\begin{array}{r|cccc} -2 & 1 & 4 & 1 & -6 \\ & & -2 & -4 & 6 \\ \hline & 1 & 2 & -3 & 0 \\ & x^2 & x & \# & R \end{array}$$

$x^3 + 4x^2 + x - 6 = (x + 2)(x^2 + 2x - 3)$

> **Factor Theorem:** $x - b$ is a factor of a polynomial $P(x)$ if and only if $P(b) = 0$. Similarly, $ax - b$ is a factor of $P(x)$ if and only if $P\left(\frac{b}{a}\right) = 0$.

Example 1: Determine if $x - 3$ and $x + 2$ are factors of $P(x) = x^3 - x^2 - 14x + 24$

$$\begin{aligned} P(3) &= (3)^3 - (3)^2 - 14(3) + 24 \\ &= 27 - 9 - 42 + 24 \\ &= 0 \end{aligned}$$

Since the remainder is 0, $x - 3$ divides evenly into $P(x)$; that means $x - 3$ is a factor of $P(x)$.

$$\begin{aligned} P(-2) &= (-2)^3 - (-2)^2 - 14(-2) + 24 \\ &= -8 - 4 + 28 + 24 \\ &= 40 \end{aligned}$$

Since the remainder is not 0, $x + 2$ does not divide evenly into $P(x)$; that means $x + 2$ is not a factor of $P(x)$.

Part 2: How to Determine a Factor of a Polynomial With Leading Coefficient 1

You could guess and check values of b that make $P(b) = 0$ until you find one that works…

Or you can use the Integral Zero Theorem to help.

> **Integral Zero Theorem:** If $x - b$ is a factor of a polynomial function $P(x)$ with leading coefficient 1 and remaining coefficients that are integers, then b is a factor of the constant term of $P(x)$.

Note: Once one of the factors of a polynomial is found, division is used to determine the other factors.

Example 2: Factor $x^3 + 2x^2 - 5x - 6$ fully.

Let $P(x) = x^3 + 2x^2 - 5x - 6$

Find a value of b such that $P(b) = 0$. Based on the factor theorem, if $P(b) = 0$, then we know that $x - b$ is a factor. We can then divide $P(x)$ by that factor.

The integral zero theorem tells us to test factors of -6.

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Test $\pm 1,\ \pm 2,\ \pm 3,$ and ± 6. Once one factor is found, you can stop testing and use that factor to divide $P(x)$.

$$\begin{aligned} P(1) &= (1)^3 + 2(1)^2 - 5(1) - 6 \\ &= 1 + 2 - 5 - 6 \\ &= -8 \end{aligned}$$

Since $P(1) \neq 0$, we know that $x - 1$ is NOT a factor of $P(x)$.

$$\begin{aligned} P(2) &= (2)^3 + 2(2)^2 - 5(2) - 6 \\ &= 8 + 8 - 10 - 6 \\ &= 0 \end{aligned}$$

Since $P(2) = 0$, we know that $x - 2$ is a factor of $P(x)$.

You can now use either long division or synthetic division to find the other factors.

Method 1: Long division

$$\begin{array}{r} x^2 + 4x + 3 \\ x - 2 \overline{)\ x^3 + 2x^2 - 5x - 6} \\ \underline{x^3 - 2x^2} \\ 4x^2 - 5x \\ \underline{4x^2 - 8x} \\ 3x - 6 \\ \underline{3x - 6} \\ R = 0 \end{array}$$

Method 2: Synthetic Division

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -5 & -6 \\ & & 2 & 8 & 6 \\ \hline & 1 & 4 & 3 & 0 \\ & x^2 & x & \# & R \end{array}$$

$x^3 + 2x^2 - 5x - 6 = (x - 2)(x^2 + 4x + 3)$

$= (x - 2)(x + 3)(x + 1)$

Example 3: Factor $x^4 + 3x^3 - 7x^2 - 27x - 18$ completely.

Let $P(x) = x^4 + 3x^3 - 7x^2 - 27x - 18$

Find a value of b such that $P(b) = 0$. Based on the factor theorem, if $P(b) = 0$, then we know that $x - b$ is a factor. We can then divide $P(x)$ by that factor.

The integral zero theorem tells us to test factors of -18.

Test $\pm 1,\ \pm 2,\ \pm 3,\ \pm 6,\ \pm 9,$ and ± 18. Once one factor is found, you can stop testing and use that factor to divide $P(x)$.

$$\begin{aligned} P(1) &= (1)^4 + 3(1)^3 - 7(1)^2 - 27(1) - 18 \\ &= -48 \end{aligned}$$

$x - 1$ is NOT a factor of $P(x)$.

$$\begin{aligned} P(-1) &= (-1)^4 + 3(-1)^3 - 7(-1)^2 - 27(-1) - 18 \\ &= 0 \end{aligned}$$

$x + 1$ IS a factor of $P(x)$.

Since $P(-1) = 0$, this tells us that $x + 1$ is a factor. Use division to determine the other factor.

$$\begin{array}{r|rrrrr} -1 & 1 & 3 & -7 & -27 & -18 \\ & & -1 & -2 & 9 & 18 \\ \hline & 1 & 2 & -9 & -18 & 0 \\ & x^3 & x^2 & x & \# & R \end{array}$$

$x^4 + 3x^3 - 7x^2 - 27x - 18 = (x + 1)(x^3 + 2x^2 - 9x - 18)$

We can now further divide $x^3 + 2x^2 - 9x - 18$ using division again or by factoring by grouping.

Method 1: Division

Test factors of -18

$$\begin{aligned} f(-2) &= (-2)^3 + 2(-2)^2 - 9(-2) - 18 \\ &= 0 \end{aligned}$$

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$x + 2$ is a factor

$$\begin{array}{r|rrrr} -2 & 1 & 2 & -9 & -18 \\ & & -2 & 0 & 18 \\ \hline & 1 & 0 & -9 & 0 \\ & x^2 & x & \# & R \end{array}$$

$x^4 + 3x^3 - 7x^2 - 27x - 18 = (x + 1)(x + 2)(x^2 - 9)$

$= (x + 1)(x + 2)(x - 3)(x + 3)$

Method 2: Factoring by Grouping

$f(x) = x^3 + 2x^2 - 9x - 18$

Group the first 2 terms and the last 2 terms and separate with an addition sign.

$f(x) = (x^3 + 2x^2) + (-9x - 18)$

Common factor within each group

$f(x) = x^2(x + 2) - 9(x + 2)$

Factor out the common binomial

$f(x) = (x + 2)(x^2 - 9)$

Therefore,

$x^4 + 3x^3 - 7x^2 - 27x - 18 = (x + 1)(x^3 + 2x^2 - 9x - 18)$

$= (x + 1)(x + 2)(x^2 - 9)$

$= (x + 1)(x + 2)(x - 3)(x + 3)$

Example 4: Try Factoring by Grouping Again

$x^4 - 6x^3 + 2x^2 - 12x$

Note: Factoring by grouping does not always work…but when it does, it saves you time!

$= (x^4 - 6x^3) + (2x^2 - 12x)$

$= x^3(x - 6) + 2x(x - 6)$

$= (x - 6)(x^3 + 2x)$

$= (x - 6)(x)(x^2 + 2)$

Part 3: How to Determine a Factor of a Polynomial With Leading Coefficient NOT 1

The integral zero theorem can be extended to include polynomials with leading coefficients that are not 1. This extension is known as the rational zero theorem.

> **Rational Zero Theorem:** Suppose $P(x)$ is a polynomial function with integer coefficients and $x = \frac{b}{a}$ is a zero of $P(x)$, where a and b are integers and $a \neq 0$. Then,
>
> - b is a factor of the constant term of $P(x)$
> - a is a factor of the leading coefficient of $P(x)$
> - $(ax - b)$ is a factor of $P(x)$

Example 5: Factor $P(x) = 3x^3 + 2x^2 - 7x + 2$

We must start by finding a value of $\frac{b}{a}$ where $P\left(\frac{b}{a}\right) = 0$.

b must be a factor of the constant term. Possible values for b are: $\pm 1, \pm 2$

a must be a factor of the leading coefficient. Possible values of a are: $\pm 1, \pm 3$

Therefore, possible values for $\frac{b}{a}$ are: $\pm 1, \pm \frac{1}{3}, \pm 2, \pm \frac{2}{3}$

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Test values of $\frac{b}{a}$ for x in $P(x)$ to find a zero.

$P(1) = 3(1)^3 + 2(1)^2 - 7(1) + 2 = 0$

Since $P(1) = 0$, $x - 1$ is a factor of $P(x)$. Use division to find the other factors.

$$\begin{array}{r|rrrr} 1 & 3 & 2 & -7 & 2 \\ & & 3 & 5 & -2 \\ \hline & 3 & 5 & -2 & 0 \\ & x^2 & x & \# & R \end{array}$$

$3x^3 + 2x^2 - 7x + 2 = (x - 1)(3x^2 + 5x - 2)$

$= (x - 1)[(3x^2 + 6x) + (-1x - 2)]$

$= (x - 1)[3x(x + 2) - 1(x + 2)]$

$= (x - 1)(x + 2)(3x - 1)$

Example 6: Factor $P(x) = 2x^3 + x^2 - 7x - 6$

Possible values for b are: $\pm 1,\ \pm 2,\ \pm 3,\ \pm 6$

Possible values of a are: $\pm 1,\ \pm 2$

Therefore, possible values for $\frac{b}{a}$ are: $\pm 1,\ \pm \frac{1}{2},\ \pm 2,\ \pm 3,\ \pm \frac{3}{2},\ \pm 6$

$f(-1) = 2(-1)^3 + (-1)^2 - 7(-1) - 6 = 0$

Therefore, $x + 1$ is a factor of $P(x)$

$$\begin{array}{r|rrrr} -1 & 2 & 1 & -7 & -6 \\ & & -2 & 1 & 6 \\ \hline & 2 & -1 & -6 & 0 \\ & x^2 & x & \# & R \end{array}$$

$2x^3 + x^2 - 7x - 6 = (x + 1)(2x^2 - x - 6)$

$= (x + 1)[(2x^2 - 4x) + (3x - 6)]$

$= (x + 1)[2x(x - 2) + 3(x - 2)]$

$= (x + 1)(x - 2)(2x + 3)$

Part 4: Application Question

Example 7: When $f(x) = 2x^3 - mx^2 + nx - 2$ is divided by $x + 1$, the remainder is -12 and $x - 2$ is a factor. Determine the values of m and n.

Hint: Use the information given to create 2 equations and then use substitution or elimination to solve.

$$\begin{aligned} f(-1) &= 2(-1)^3 - m(-1)^2 + n(-1) - 2 \\ -12 &= -2 - m - n - 2 \\ -8 &= -m - n \end{aligned}$$

Equation (1): $-8 = -m - n$

$$\begin{aligned} f(2) &= 2(2)^3 - m(2)^2 + n(2) - 2 \\ 0 &= 16 - 4m + 2n - 2 \\ -14 &= -4m + 2n \end{aligned}$$

Equation (2): $-14 = -4m + 2n$

Multiply equation (1) by 2: $-16 = -2m - 2n$

Add this to equation (2): $-14 = -4m + 2n$

$$\begin{aligned} -30 &= -6m \\ 5 &= m \end{aligned}$$

Substitute $m = 5$ into equation (1) or (2):

$$\begin{aligned} -8 &= -5 - n \\ n &= 3 \end{aligned}$$

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So $m = 5$ and $n = 3$

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