

3.4 Solve Rational Equations and Inequalities Lesson with solutions

This document was composed and formatted by Annie Bernatchez.

The course material originated from www.jensenmath.ca

Part 1: Rational Expressions

Rational Expression: the quotient of two polynomials, $\frac{p(x)}{q(x)}$, where $q(x) \neq 0$

Example 1: Simplify and state the restrictions of each rational expression

a) $\frac{2x^2 - 8}{x^2 + 3x + 2}$

$$= \frac{2(x^2 - 4)}{(x+2)(x+1)} = \frac{2(x-2)(x+2)}{(x+2)(x+1)} = \frac{2(x-2)}{x+1}; x \neq -2, -1$$

b) $\frac{x^3 - x^2 - x + 1}{3x^3 - 3}$

$$= \frac{(x^3 - x^2) + (-x + 1)}{3(x^3 - 1)} = \frac{x^2(x-1) - 1(x-1)}{3(x-1)(x^2 + x + 1)} = \frac{(x-1)(x^2 - 1)}{3(x-1)(x^2 + x + 1)} = \frac{(x-1)(x+1)}{3(x^2 + x + 1)}; x \neq 1$$

Part 2: Solve Rational Equations

Steps for solving rational equations:

1. Fully factor both sides of the equation
2. Multiply both sides by a common denominator (cross multiply if appropriate)
3. Continue to solve as you would a normal polynomial equation
4. State restrictions throughout (values of x that would make denominator equal zero)

Example 2: Solve each equation

a) $\frac{4}{3x-5} = 4$

$$(3x-5)\left(\frac{4}{3x-5}\right) = 4(3x-5); x \neq \frac{5}{3}$$

$$4 = 4(3x-5)$$

$$4 = 12x - 20$$

$$24 = 12x$$

$$x = 2$$

b) $\frac{6}{x-2} = x-1$

$$(x-2)\left(\frac{6}{x-2}\right) = (x-2)(x-1); x \neq 2$$

$$6 = x^2 - x - 2x + 2$$

$$0 = x^2 - 3x - 4$$

$$0 = (x-4)(x+1)$$

$$x-4=0 \Rightarrow x_1=4; x+1=0 \Rightarrow x_2=-1$$

c) $\frac{x-5}{x^2-3x-4} = \frac{3x+2}{x^2-1}$

$$\frac{x-5}{(x-4)(x+1)} = \frac{3x+2}{(x-1)(x+1)}; x \neq -1, 1, 4$$

$$(x-1)(x+1)(x-4)(x-5) = (x-4)(x+1)(x-1)(3x+2) \text{ — cancel the shared } (x+1) \text{ factor from both sides:}$$

$$(x-1)(x-5) = (x-4)(3x+2)$$

$$x^2 - 6x + 5 = 3x^2 - 10x - 8$$

$$0 = 2x^2 - 4x - 13 \text{ — not factorable, use the quadratic formula}$$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(2)(-13)}}{2(2)} = \frac{4 \pm \sqrt{120}}{4} = \frac{4 \pm 2\sqrt{30}}{4} = \frac{2(2 \pm \sqrt{30})}{4} = \frac{2 \pm \sqrt{30}}{2}$$

Part 3: Solve Rational Inequalities

REMEMBER: Solving inequalities is the same as solving equations. However, when both sides of an inequality are multiplied or divided by a negative number, the inequality sign must be reversed.

Steps for solving rational inequalities algebraically:

1. Use inverse operations to move all terms to one side of the inequality
2. Combine the expressions by using a common denominator
3. Factor the polynomial
4. Find the interval(s) where the function is positive or negative by making a factor table

To make a factor table:

- Use x -intercepts and vertical asymptotes to divide into intervals
- Use a test point within each interval to find the sign of each factor
- Determine the overall sign of the product by multiplying signs of each factor within each interval.

Example 3: Solve each inequality algebraically

a) $\frac{x^2 + 6x + 5}{2x^2 - 7x + 3} < 0$

$$\frac{(x+5)(x+1)}{(2x-1)(x-3)} < 0; \text{zeros: } x = -5, -1; \text{restrictions: } x \neq \frac{1}{2}, 3$$

interval	$x < -5$	$-5 < x < -1$	$-1 < x < 0.5$	$0.5 < x < 3$	$x > 3$
test point	-6	-2	0	1	4
$x+5$	-	+	+	+	+
$x+1$	-	-	+	+	+
$2x-1$	-	-	-	+	+
$x-3$	-	-	-	-	+
overall	+	-	+	-	+

The inequality is true when $-5 < x < -1$ or $0.5 < x < 3$, i.e. $x \in (-5, -1) \cup (0.5, 3)$

b) $x - 2 < \frac{8}{x}$

$$x - 2 - \frac{8}{x} < 0 \Rightarrow \frac{x^2 - 2x - 8}{x} < 0 \Rightarrow \frac{(x-4)(x+2)}{x} < 0; \text{zeros: } x = -2, 4; \text{restriction: } x \neq 0$$

interval	$x < -2$	$-2 < x < 0$	$0 < x < 4$	$x > 4$
test point	-3	-1	1	5
$x-4$	-	-	-	+
$x+2$	-	+	+	+
x	-	-	+	+
overall	-	+	-	+

The inequality is true when $x < -2$ or $0 < x < 4$, i.e. $x \in (-\infty, -2) \cup (0, 4)$

c) $\frac{x+3}{x+1} \geq \frac{x-2}{x-3}$

$$\frac{(x+3)(x-3) - (x-2)(x+1)}{(x+1)(x-3)} \geq 0$$

3.4 Solve Rational Equations and Inequalities Lesson with solutions

$$\frac{x^2 - 9 - (x^2 - x - 2)}{(x+1)(x-3)} \geq 0 \Rightarrow \frac{x-7}{(x+1)(x-3)} \geq 0; \text{zero: } x = 7; \text{restrictions: } x \neq -1, 3$$

interval	$x < -1$	$-1 < x < 3$	$3 < x < 7$	$x > 7$
test point	-2	0	4	8
$x - 7$	-	-	-	+
$x + 1$	-	+	+	+
$x - 3$	-	-	+	+
overall	-	+	-	+

The inequality is true when $-1 < x < 3$ or $x \geq 7$, i.e. $x \in (-1, 3) \cup [7, \infty)$

d) $\frac{x^3 + 6x^2 - 2x}{x^2 + 4} \geq 2$

$$\frac{x^3 + 6x^2 - 2x - 2(x^2 + 4)}{x^2 + 4} \geq 0 \Rightarrow \frac{x^3 + 4x^2 - 2x - 8}{x^2 + 4} \geq 0$$

$$\frac{x^2(x+4) - 2(x+4)}{x^2 + 4} \geq 0 \Rightarrow \frac{(x+4)(x^2 - 2)}{x^2 + 4} \geq 0; \text{zeros: } x = -4, \pm\sqrt{2}; \text{no restrictions } (x^2 + 4 \text{ is never zero})$$

interval	$x < -4$	$-4 < x < -\sqrt{2}$	$-\sqrt{2} < x < \sqrt{2}$	$x > \sqrt{2}$
test point	-5	-3	0	2
$x + 4$	-	+	+	+
$x^2 - 2$	+	+	-	+
$x^2 + 4$	+	+	+	+
overall	-	+	-	+

The inequality is true when $-4 \leq x \leq -\sqrt{2}$ or $x \geq \sqrt{2}$, i.e. $x \in [-4, -\sqrt{2}] \cup [\sqrt{2}, \infty)$