

5.2 Compound Angle Formulas Lesson with solutions

This document was composed and formatted by Annie Bernatchez.

The course material originated from www.jensenmath.ca

Compound angle: an angle that is created by adding or subtracting two or more angles.

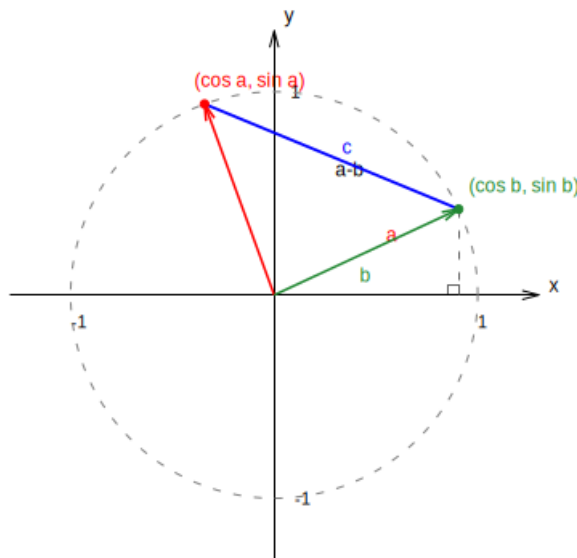
Part 1: Proof of $\cos(x - y)$

Normal algebra rules do not apply:

$$\cos(x - y) \neq \cos x - \cos y$$

So what does $\cos(x - y) = ?$

Consider the diagram to the right...



By the cosine law:

$$c^2 = 1^2 + 1^2 - 2(1)(1)\cos(a - b)$$

$$c^2 = 2 - 2\cos(a - b) \text{ THIS IS EQUATION 1}$$

But notice that c has endpoints of $(\cos a, \sin a)$ and $(\cos b, \sin b)$

Using the distance formula $distance = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$c = \sqrt{(\cos a - \cos b)^2 + (\sin a - \sin b)^2}$$

$$c^2 = (\cos a - \cos b)^2 + (\sin a - \sin b)^2$$

$$c^2 = \cos^2 a - 2\cos a \cos b + \cos^2 b + \sin^2 a - 2\sin a \sin b + \sin^2 b$$

$$c^2 = 1 - 2\cos a \cos b - 2\sin a \sin b + 1$$

$$c^2 = 2 - 2\cos a \cos b - 2\sin a \sin b \text{ THIS IS EQUATION 2}$$

Set equations 1 and 2 equal

$$2 - 2\cos(a - b) = 2 - 2\cos a \cos b - 2\sin a \sin b$$

$$-2\cos(a - b) = -2\cos a \cos b - 2\sin a \sin b$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

Replacing b with $-y$ (and a with x), and using the even/odd properties $\cos(-y) = \cos y$ and $\sin(-y) = -\sin y$:

$$\cos(x+y) = \cos x \cos(-y) + \sin x \sin(-y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

Part 2: Proofs of other compound angle formulas

Table 1. Even/Odd Properties

$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
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Example 1: Prove $\cos(x-y) = \cos x \cos y + \sin x \sin y$

LS	RS
$= \cos(x-y)$ $= \cos[x + (-y)]$ $= \cos x \cos(-y) - \sin x \sin(-y)$ $= \cos x \cos y - \sin x(-\sin y)$ $= \cos x \cos y + \sin x \sin y$	$= \cos x \cos y + \sin x \sin y$

LS = RS

Example 2: Prove addition and subtraction formulas for sine using co-function identities and the subtraction formula for cosine.

Table 2. Co-Function Identities

$\cos\left(\frac{\pi}{2} - x\right) = \sin x$	$\sin\left(\frac{\pi}{2} - x\right) = \cos x$
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a) Prove $\sin(x+y) = \sin x \cos y + \cos x \sin y$

LS	RS
$= \sin(x+y)$ $= \cos\left(\frac{\pi}{2} - (x+y)\right)$ $= \cos\left(\left(\frac{\pi}{2} - x\right) - y\right)$ $= \cos\left(\frac{\pi}{2} - x\right)\cos y + \sin\left(\frac{\pi}{2} - x\right)\sin y$ $= \sin x \cos y + \cos x \sin y$	$= \sin x \cos y + \cos x \sin y$

LS = RS

b) Prove $\sin(x-y) = \sin x \cos y - \cos x \sin y$

LS	RS
$= \sin(x-y)$ $= \sin x \cos(-y) + \cos x \sin(-y)$ $= \sin x \cos y + \cos x(-\sin y)$ $= \sin x \cos y - \cos x \sin y$	$= \sin x \cos y - \cos x \sin y$

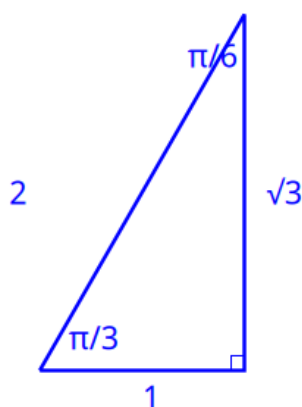
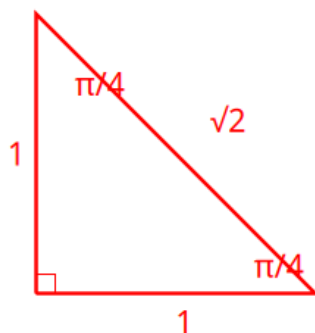
LS = RS

Table 3. Compound Angle Formulas

$\sin(x+y) = \sin x \cos y + \cos x \sin y$	$\sin(x-y) = \sin x \cos y - \cos x \sin y$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$	$\cos(x-y) = \cos x \cos y + \sin x \sin y$
$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$	$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Part 3: Determine Exact Trig Ratios for Angles other than Special Angles

By expressing an angle as a sum or difference of angles in the special triangles, exact values of other angles can be determined.



Example 3: Use compound angle formulas to determine exact values for

a) $\sin \frac{\pi}{12}$

$$\begin{aligned}
 \sin \frac{\pi}{12} &= \sin \left(\frac{4\pi}{12} - \frac{3\pi}{12} \right) \\
 &= \sin \left(\frac{\pi}{3} - \frac{\pi}{4} \right) \\
 &= \sin \left(\frac{\pi}{3} \right) \cos \left(\frac{\pi}{4} \right) - \cos \left(\frac{\pi}{3} \right) \sin \left(\frac{\pi}{4} \right) \\
 &= \frac{\sqrt{3}}{2} \left(\frac{1}{\sqrt{2}} \right) - \left(\frac{1}{2} \right) \left(\frac{1}{\sqrt{2}} \right) \\
 &= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \\
 &= \frac{\sqrt{3} - 1}{2\sqrt{2}}
 \end{aligned}$$

b) $\tan \left(-\frac{5\pi}{12} \right)$

$$\tan \left(-\frac{5\pi}{12} \right) = -\tan \left(\frac{5\pi}{12} \right)$$

$$\begin{aligned} &= -\tan\left(\frac{2\pi}{12} + \frac{3\pi}{12}\right) \\ &= -\frac{\tan\left(\frac{2\pi}{12}\right) + \tan\left(\frac{3\pi}{12}\right)}{1 - \tan\left(\frac{2\pi}{12}\right)\tan\left(\frac{3\pi}{12}\right)} \\ &= -\frac{\tan\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \tan\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{4}\right)} \\ &= -\frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}} \\ &= -\frac{\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}}} \\ &= -\frac{\frac{1 + \sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3} - 1}{\sqrt{3}}} \\ &= -\frac{1 + \sqrt{3}}{\sqrt{3} - 1} \end{aligned}$$

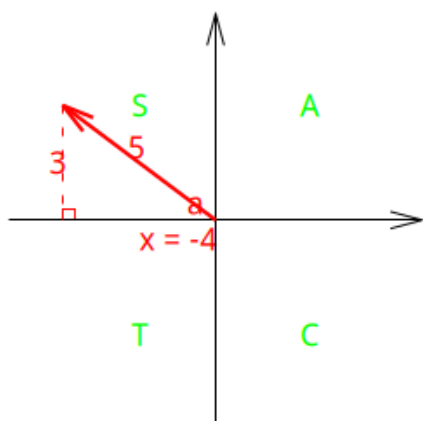
Part 4: Use Compound Angle Formulas to Simplify Trig Expressions

Example 4: Simplify the following expression

$$\begin{aligned} &\cos \frac{7\pi}{12} \cos \frac{5\pi}{12} + \sin \frac{7\pi}{12} \sin \frac{5\pi}{12} \\ &= \cos\left(\frac{7\pi}{12} - \frac{5\pi}{12}\right) \\ &= \cos \frac{2\pi}{12} \\ &= \cos \frac{\pi}{6} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

Part 5: Application

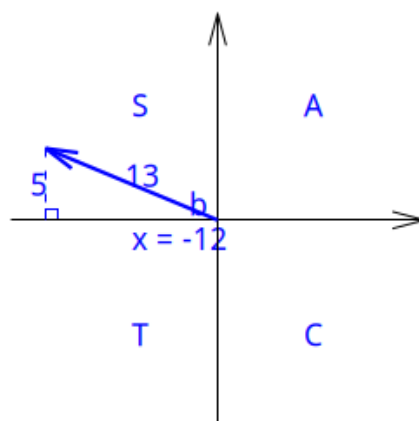
Example 5: Evaluate $\sin(a + b)$, where a and b are both angles in the second quadrant; given $\sin a = \frac{3}{5}$ and $\sin b = \frac{5}{13}$
Start by drawing both terminal arms in the second quadrant and solving for the third side.



$$x^2 + 3^2 = 5^2$$

$$x^2 = 16$$

$$x = 4$$



$$x^2 + 5^2 = 13^2$$

$$x^2 = 144$$

$$x = 12$$

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

$$= \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) + \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right)$$

$$= -\frac{36}{65} - \frac{20}{65}$$

$$= -\frac{56}{65}$$