

5.1 Co-function Identities Lesson with solutions

This document was composed and formatted by Annie Bernatchez.

The course material originated from www.jensenmath.ca

Part 1: Remembering How to Prove Trig Identities

Table 1. Fundamental Trigonometric Identities

Reciprocal Identities	Quotient Identities	Pythagorean Identities
$\csc \theta = \frac{1}{\sin \theta}$ $\sec \theta = \frac{1}{\cos \theta}$ $\cot \theta = \frac{1}{\tan \theta}$	$\frac{\sin \theta}{\cos \theta} = \tan \theta$ $\frac{\cos \theta}{\sin \theta} = \cot \theta$	$\sin^2 \theta + \cos^2 \theta = 1$

Table 2. Tips and Tricks

Reciprocal Identities	Quotient Identities	Pythagorean Identities
Square both sides $\csc^2 \theta = \frac{1}{\sin^2 \theta}$ $\sec^2 \theta = \frac{1}{\cos^2 \theta}$ $\cot^2 \theta = \frac{1}{\tan^2 \theta}$	Square both sides $\frac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta$ $\frac{\cos^2 \theta}{\sin^2 \theta} = \cot^2 \theta$	Rearrange the identity $\sin^2 \theta = 1 - \cos^2 \theta$ $\cos^2 \theta = 1 - \sin^2 \theta$ Divide by either $\sin^2 \theta$ or $\cos^2 \theta$ $1 + \cot^2 \theta = \csc^2 \theta$ $\tan^2 \theta + 1 = \sec^2 \theta$

General tips for proving identities:

- Separate into LS and RS. Terms may NOT cross between sides.
- Try to change everything to $\sin \theta$ or $\cos \theta$
- If you have two fractions being added or subtracted, find a common denominator and combine the fractions.
- Use difference of squares $\rightarrow 1 - \sin^2 \theta = (1 - \sin \theta)(1 + \sin \theta)$
- Use the power rule $\rightarrow \sin^6 \theta = (\sin^2 \theta)^3$

Example 1: Prove each of the following identities

a) $\tan^2 x + 1 = \sec^2 x$

LS	RS
$= \tan^2 x + 1$ $= \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}$ $= \frac{\sin^2 x + \cos^2 x}{\cos^2 x}$ $= \frac{1}{\cos^2 x}$	$= \sec^2 x$ $= \frac{1}{\cos^2 x}$

LS = RS

b) $\cos^2 x = (1 - \sin x)(1 + \sin x)$

LS	RS
$= \cos^2 x$	$= (1 - \sin x)(1 + \sin x)$ $= 1 - \sin^2 x$ $= \cos^2 x$

LS = RS

c) $\frac{\sin^2 x}{1 - \cos x} = 1 + \cos x$

LS	RS
$\begin{aligned} &= \frac{\sin^2 x}{1 - \cos x} \\ &= \frac{1 - \cos^2 x}{1 - \cos x} \\ &= \frac{(1 - \cos x)(1 + \cos x)}{1 - \cos x} \\ &= 1 + \cos x \end{aligned}$	$= 1 + \cos x$

LS = RS

Part 2: Transformation Identities

Because of their periodic nature, there are many equivalent trigonometric expressions.

Horizontal translations of $\pi/2$ that involve both a sine function and a cosine function can be used to obtain two equivalent functions with the same graph.

Translating the cosine function $\pi/2$ to the **right**, $f(x) = \cos\left(x - \frac{\pi}{2}\right)$ results in the graph of the sine function, $f(x) = \sin x$.

Similarly, translating the sine function $\pi/2$ to the **left**, $f(x) = \sin\left(x + \frac{\pi}{2}\right)$ results in the graph of the cosine function, $f(x) = \cos x$.

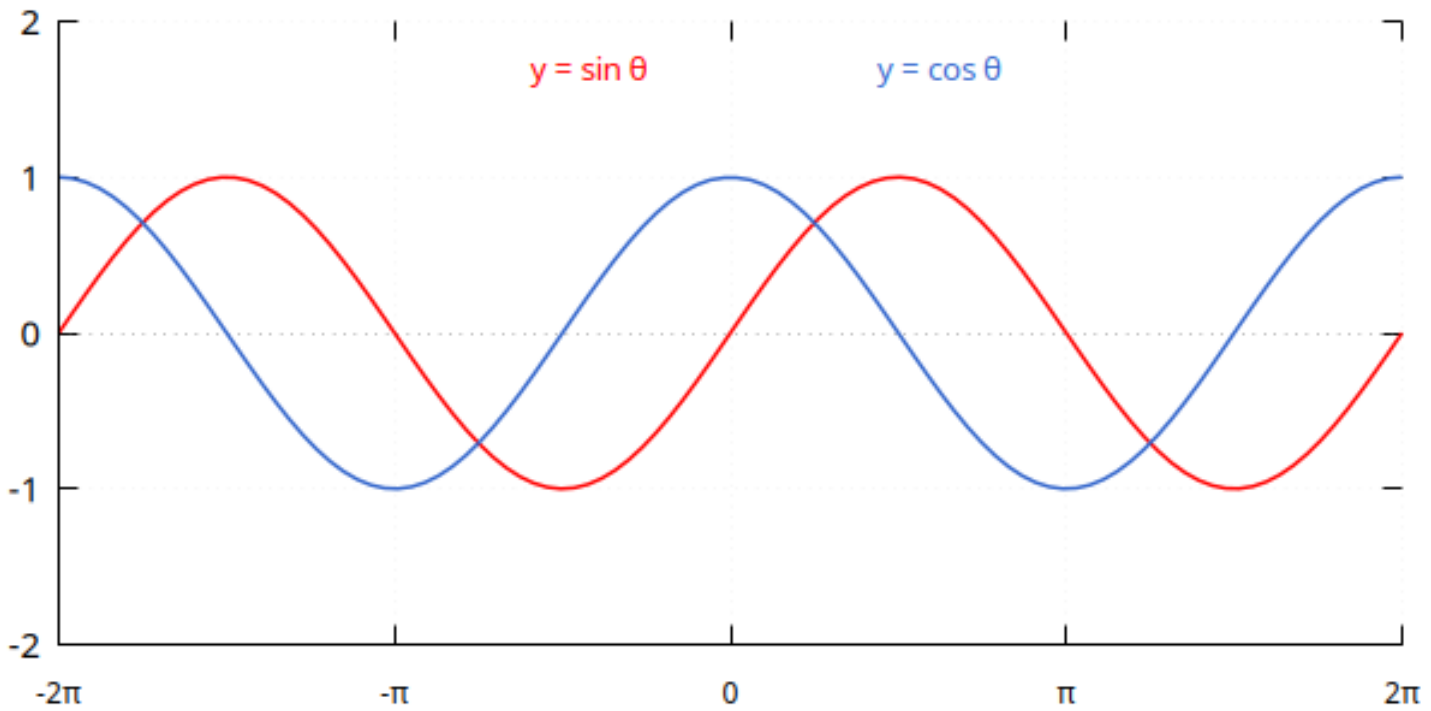
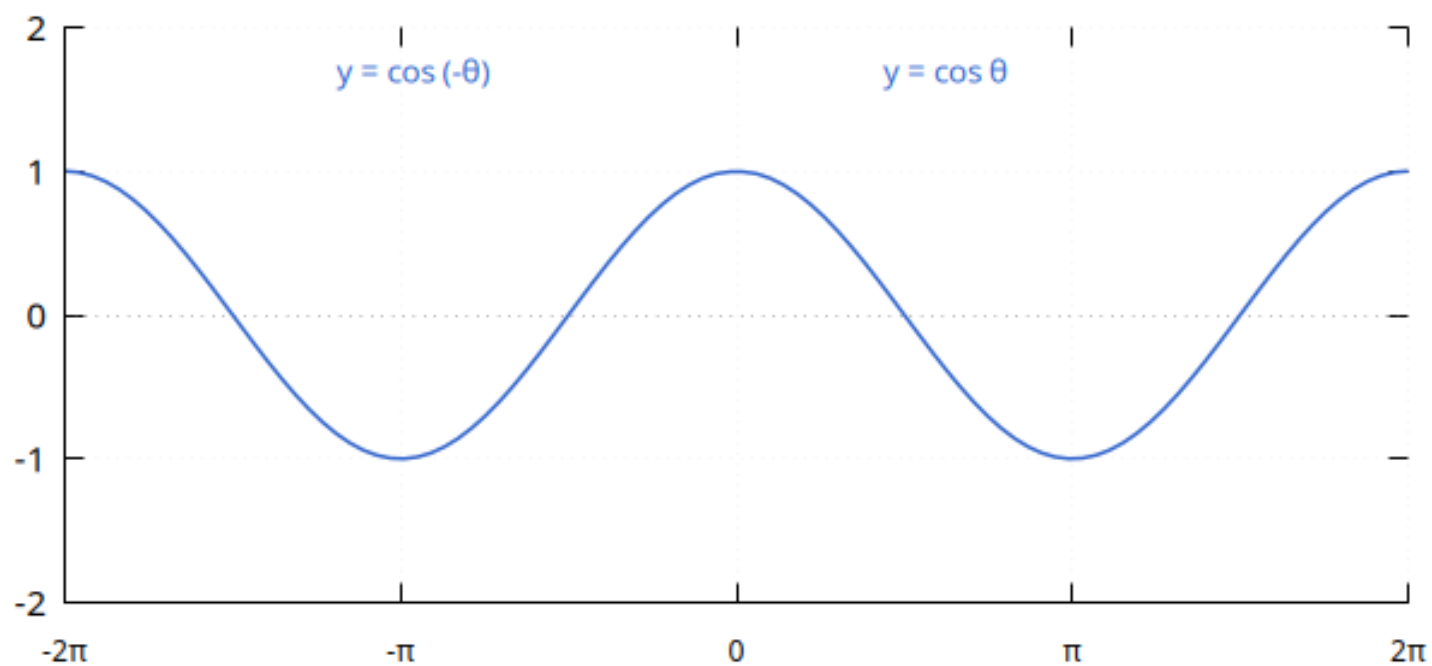


Table 3. Transformation Identities

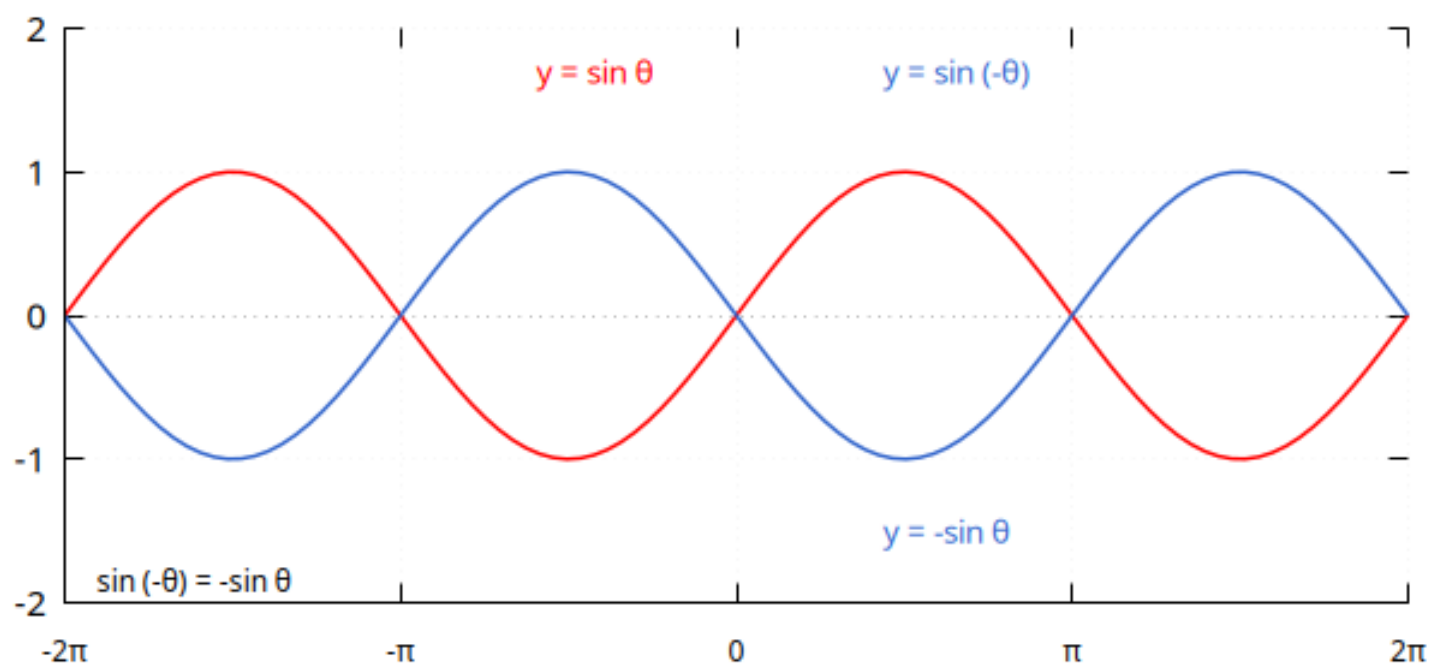
$\cos\left(x - \frac{\pi}{2}\right) = \sin x$	$\sin\left(x + \frac{\pi}{2}\right) = \cos x$
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Part 3: Even/Odd Function Identities

Remember that $\cos x$ is an **even** function. Reflecting its graph across the y-axis results in two equivalent functions with the same graph.



$\sin x$ and $\tan x$ are both **odd** functions. They have rotational symmetry about the origin.



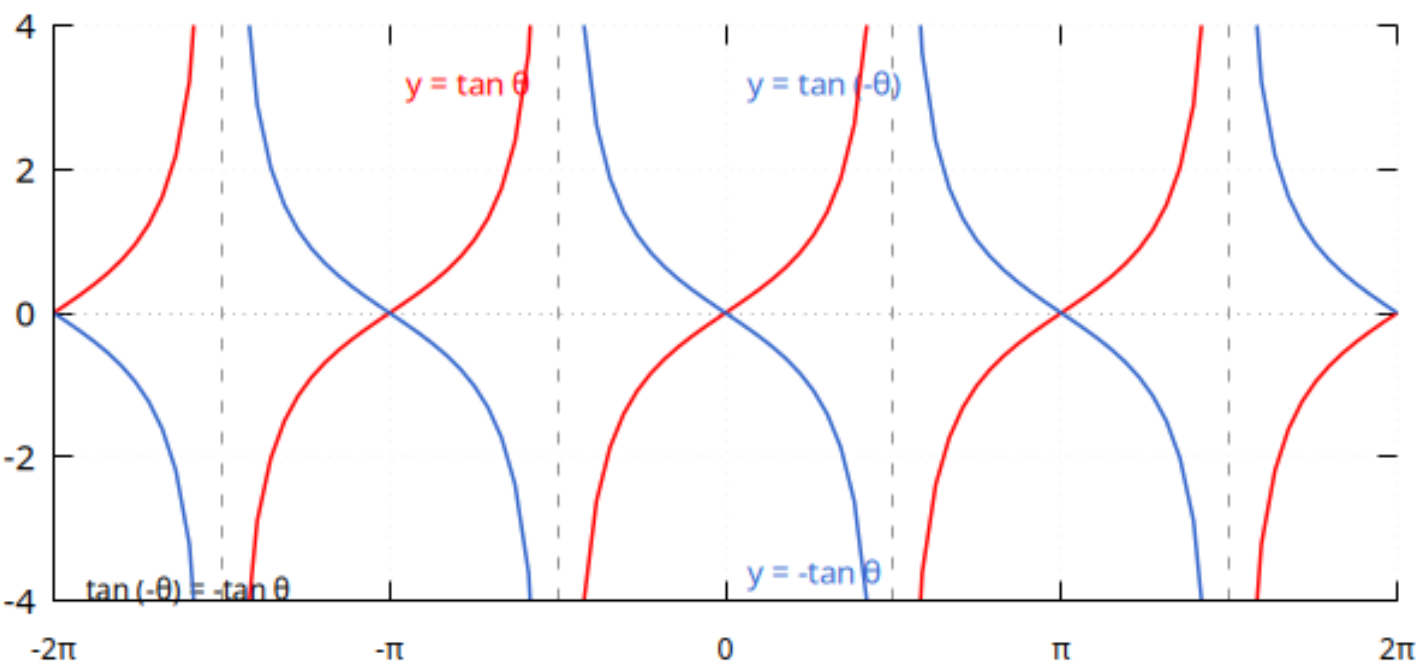


Table 4. Even/Odd Identities

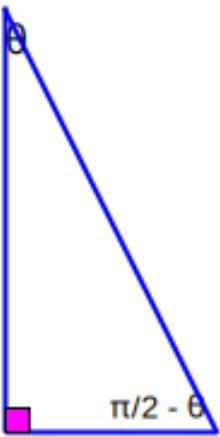
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$	$\tan(-x) = -\tan x$
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Part 4: Co-function Identities

The co-function identities describe trigonometric relationships between complementary angles in a right triangle.

Table 5. Co-Function Identities

$\cos\left(\frac{\pi}{2} - x\right) = \sin x$	$\sin\left(\frac{\pi}{2} - x\right) = \cos x$
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We could identify other equivalent trigonometric expressions by comparing principle angles drawn in standard position in quadrants II, III, and IV with their related acute (reference) angle in quadrant I.

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Principle in Quadrant II	Principle in Quadrant III	Principle in Quadrant IV
$\sin(\pi - x) = \sin x$	$\sin(\pi + x) = -\sin x$	$\sin(2\pi - x) = -\sin x$

Example 2: Prove both co-function identities using transformation identities

a) $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

LS	RS
$= \cos\left(\frac{\pi}{2} - x\right)$ $= \cos\left(-x + \frac{\pi}{2}\right)$ $= \cos\left[-\left(x - \frac{\pi}{2}\right)\right]$ $= \cos\left(x - \frac{\pi}{2}\right)$ $= \sin x$	$= \sin x$

LS = RS

b) $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

LS	RS
$= \sin\left(\frac{\pi}{2} - x\right)$	$= \cos x$ $= \cos(-x)$ $= \sin\left(-x + \frac{\pi}{2}\right)$ $= \sin\left(\frac{\pi}{2} - x\right)$

LS = RS

Part 5: Apply the Identities

Example 3: Given that $\sin \frac{\pi}{5} \cong 0.5878$, use equivalent trigonometric expressions to evaluate the following:

a) $\cos \frac{3\pi}{10}$

$$= \sin\left(\frac{\pi}{2} - \frac{3\pi}{10}\right)$$

$$= \sin\left(\frac{5\pi}{10} - \frac{3\pi}{10}\right)$$

$$= \sin\left(\frac{2\pi}{10}\right)$$

$$= \sin\left(\frac{\pi}{5}\right)$$

$$\cong 0.5878$$

b) $\cos \frac{7\pi}{10}$

$$= \sin\left(\frac{\pi}{2} - \frac{7\pi}{10}\right)$$

$$= \sin\left(\frac{5\pi}{10} - \frac{7\pi}{10}\right)$$

$$= \sin\left(-\frac{2\pi}{10}\right)$$

$$= -\sin\left(\frac{\pi}{5}\right)$$

$$\cong -0.5878$$