

3.8 The Natural Logarithm Lesson with solutions

This document was composed and formatted by Annie Bernatchez.

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Part 1: What is 'e'?

Example 1: Suppose you invest \$1 at 100% interest for 1 year at various compounding levels. What is the highest amount of money you can have after 1 year?

Note: the formula used for compound interest of \$1 at 100% interest annually compounded n times during the year is:

$$A = 1\left(1 + \frac{1}{n}\right)^n$$

Compounding Level, n	Amount, A in dollars
Annually (once a year)	$A = 1\left(1 + \frac{1}{1}\right)^1 = 2$
Semi-annually (2-times)	$A = 1\left(1 + \frac{1}{2}\right)^2 = 2.25$
Quarterly (4-times)	$A = 1\left(1 + \frac{1}{4}\right)^4 = 2.4414$
Monthly (12-times)	$A = 1\left(1 + \frac{1}{12}\right)^{12} = 2.61304$
Daily (365-times)	$A = 1\left(1 + \frac{1}{365}\right)^{365} = 2.71457$
Secondly (31 536 000-times)	$A = 1\left(1 + \frac{1}{31\,536\,000}\right)^{31\,536\,000} = 2.718281785$
Continuously (1 000 000 000-times)	$A = 1\left(1 + \frac{1}{1\,000\,000\,000}\right)^{1\,000\,000\,000} = 2.718281827$

Properties of e :

- $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \cong 2.718\,281\,828\,459$
- e is an **irrational** number, similar to π . They are non-terminating and non-repeating.
- $\log_e x$ is known as the **natural logarithm** and can be written as **ln x**
- Many naturally occurring phenomena can be modelled using base- e exponential and logarithmic functions.
- $\log_e e = \ln e = 1$

Part 2: Reminder of Log Rules

Rule	Formula
Power Law of Logarithms	$\log_b x^n = n \log_b x$ for $b > 0, b \neq 1, x > 0$
Product Law of Logarithms	$\log_b(mn) = \log_b m + \log_b n$ for $b > 0, b \neq 1, m > 0, n > 0$
Quotient Law of Logarithms	$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$ for $b > 0, b \neq 1, m > 0, n > 0$
Change of Base Formula	$\log_b m = \frac{\log m}{\log b}, m > 0, b > 0, b \neq 1$
Exponential to Logarithmic	$y = b^x \rightarrow x = \log_b y$
Logarithmic to Exponential	$y = \log_b x \rightarrow x = b^y$

Other useful tips

$$\begin{aligned}\log_a(a^b) &= b \\ \log a &= \log_{10} a \\ \log_b b &= 1\end{aligned}$$

Part 3: Solving Problems Involving e

Example 2: Evaluate each of the following

a) e^3

$$\cong 20.086$$

b) $\ln 10$

$$\cong 2.303$$

c) $\ln e$

$$= 1$$

Example 3: Solve each of the following equations

a) $20 = 3e^x$

$$\begin{aligned}\frac{20}{3} &= e^x \\ \ln\left(\frac{20}{3}\right) &= \ln(e^x) \\ \ln\left(\frac{20}{3}\right) &= x \cdot \ln(e) \\ \ln\left(\frac{20}{3}\right) &= x(1) \\ x &\cong 1.897\end{aligned}$$

b) $e^{1-2x} = 55$

$$\begin{aligned}\ln(e)^{1-2x} &= \ln(55) \\ (1-2x)(\ln(e)) &= \ln(55) \\ (1-2x)(1) &= \ln(55) \\ 1-2x &= \ln(55) \\ 1 - \ln(55) &= 2x \\ x &\cong -1.504\end{aligned}$$

c) $2\ln(x-3) - 7 = 3$

$$\begin{aligned}2\ln(x-3) &= 10 \\ \ln(x-3) &= 5 \\ e^5 &= x-3 \\ x &= e^5 + 3 \\ x &\cong 151.413\end{aligned}$$

d) $\ln(4e^x) = 2$

$$\begin{aligned}e^2 &= 4e^x \\ \frac{e^2}{4} &= e^x \\ \ln\left(\frac{e^2}{4}\right) &= \ln(e^x) \\ \ln\left(\frac{e^2}{4}\right) &= x \cdot \ln(e) \\ x &\cong 0.614\end{aligned}$$

Part 4: Graphing Functions Involving e

Example 4: Graph the functions $y = e^x$ and $y = \ln x$

Note: $y = \ln x$ is the inverse of $y = e^x$

Table 1. $y = e^x$

x	y
-1	0.37
0	1

1	2.72
HA	$y = 0$

Table 2. $y = \ln x$

x	y
0.37	-1
1	0
2.72	1
VA	$x = 0$

