

3.1 Introduction to Logarithms and Review of Exponentials Lesson with solutions

This document was composed and formatted by Annie Bernatchez.

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In this section you will learn about how a logarithmic function is the inverse of an exponential function. You will also learn how to express exponential equations in logarithmic form.

Part 1: Review of Exponential Functions

Equation: $y = a(b)^x$

- a = initial amount
- b = growth ($b > 1$) or decay ($0 < b < 1$) factor
- y = future amount
- x = number of times a has increased or decreased

To calculate x , use the equation:

$$x = \frac{\text{total time}}{\text{time it takes for one growth or decay period}}$$

Example 1: An insect colony has a current population of 50 insects. Its population doubles every 3 days.

a) What is the population after 12 days?

$$\begin{aligned}y &= 50(2)^{\frac{12}{3}} \\y &= 50(2)^4 \\y &= 800\end{aligned}$$

b) How long until the population reaches 25 600?

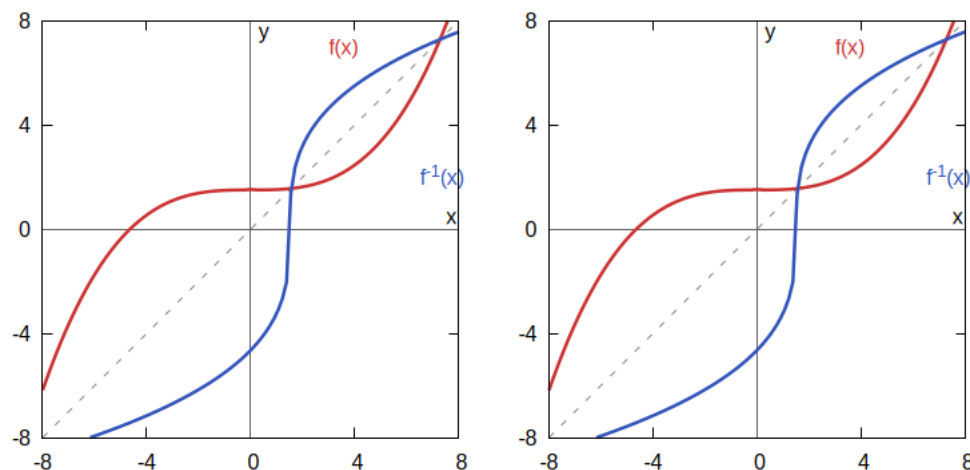
$$\begin{aligned}25\,600 &= 50(2)^{\frac{t}{3}} \\512 &= 2^{\frac{t}{3}} \\\log 512 &= \log 2^{\frac{t}{3}} \\\log 512 &= \left(\frac{t}{3}\right) \log 2 \\\frac{\log 512}{\log 2} &= \frac{t}{3} \\9 &= \frac{t}{3} \\t &= 27 \text{ days}\end{aligned}$$

Part 2: Review of Inverse Functions

Inverse of a function:

- The inverse of a function f is denoted as f^{-1}
- The function and its inverse have the property that if $f(a) = b$, then $f^{-1}(b) = a$
- So if $f(5) = 13$, then $f^{-1}(13) = 5$
- More simply put: the inverse of a function has all the same points as the original function, except that the x 's and y 's have been reversed.

The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected in the line $y = x$. This is true for all functions and their inverses.



Example 2: Determine the equation of the inverse of the function $f(x) = 3(x - 5)^2 + 1$

Method for finding the inverse algebraically:

1. Replace $f(x)$ with y
2. Switch the x and y variables
3. Isolate for y
4. Replace y with $f^{-1}(x)$

$$\begin{aligned}
 y &= 3(x - 5)^2 + 1 \\
 x &= 3(y - 5)^2 + 1 \\
 \frac{x - 1}{3} &= (y - 5)^2 \\
 \pm \sqrt{\frac{x - 1}{3}} &= y - 5 \\
 5 \pm \sqrt{\frac{x - 1}{3}} &= y
 \end{aligned}$$

Equation of inverse: $f^{-1}(x) = 5 \pm \sqrt{\frac{x - 1}{3}}$

Part 3: Review of Exponent Laws

Name	Rule
Product Rule	$x^a \cdot x^b = x^{a+b}$
Quotient Rule	$\frac{x^a}{x^b} = x^{a-b}$
Power of a Power Rule	$(x^a)^b = x^{a \times b}$
Negative Exponent Rule	$x^{-a} = \frac{1}{x^a}$
Exponent of Zero	$x^0 = 1$

Part 4: Inverse of an Exponential Function

Example 3:

a) Find the equation of the inverse of $f(x) = 2^x$.

$$\begin{aligned} y &= 2^x \\ x &= 2^y \\ \log x &= \log 2^y \\ \log x &= y \log 2 \\ y &= \frac{\log x}{\log 2} \\ y &= \log_2 x \end{aligned}$$

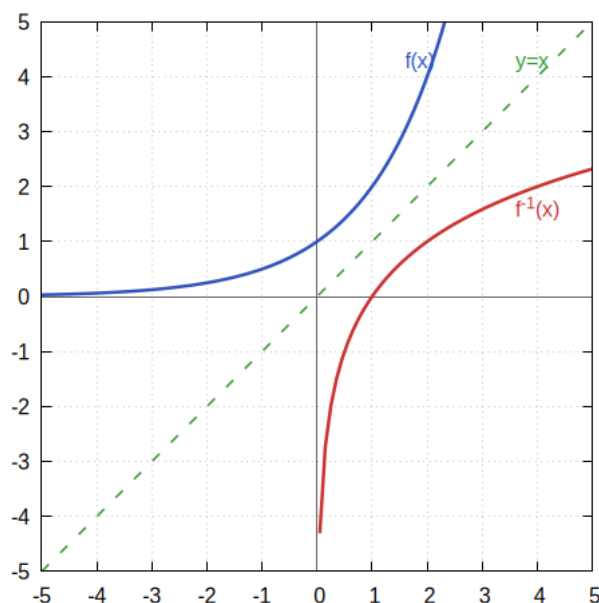
$$f^{-1}(x) = \log_2 x$$

Note: this step uses the 'change of base' formula that we will cover later in the unit: $\log_b m = \frac{\log m}{\log b}$

b) Graph both $f(x)$ and $f^{-1}(x)$.

Note: just swap the x and y coordinates to get key points for the inverse of a function. The graph should appear to be a reflection across the line $y = x$.

$f(x) = 2^x$		$f^{-1}(x) = \log_2 x$	
x	y	x	y
-2	0.25	0.25	-2
-1	0.5	0.5	-1
0	1	1	0
1	2	2	1
2	4	4	2



c) Complete the chart of key properties for both functions

$y = 2^x$	$y = \log_2 x$
x -int: none	x -int: (1, 0)
y -int: (0, 1)	y -int: none
Domain: $\{x \in \mathbb{R}\}$	Domain: $\{x \in \mathbb{R} \mid x > 0\}$
Range: $\{y \in \mathbb{R} \mid y > 0\}$	Range: $\{y \in \mathbb{R}\}$
Asymptote: horizontal asymptote at $y = 0$	Asymptote: vertical asymptote at $x = 0$

Part 5: What is a Logarithmic Function?

The logarithmic function is the **inverse** of the exponential function with the same base.

The **logarithmic function** is defined as $y = \log_b x$, or y equals the logarithm of x to the base b .

The function is defined only for $b > 0, b \neq 1$

In this notation, y is the exponent to which the base, b , must be raised to give the value of x .

In other words, the solution to a logarithm is always an **exponent**.

The logarithmic function is most useful for solving for unknown **exponents**.

Common logarithms are logarithms with a base of 10. It is not necessary to write the base for common logarithms:

$\log x$ means the same as $\log_{10} x$

Part 6: Writing Equivalent Exponential and Logarithmic Expressions

Exponential equations can be written in logarithmic form, and vice versa:

$$\bullet y = b^x \rightarrow x = \log_b y$$

$$\bullet y = \log_b x \rightarrow x = b^y$$

Example 4: Rewrite each equation in logarithmic form

a) $16 = 2^4$

$$\log_2 16 = 4$$

b) $m = n^3$

$$\log_n m = 3$$

c) $3^{-2} = \frac{1}{9}$

$$\log_3 \left(\frac{1}{9} \right) = -2$$

Example 5: Write each logarithmic equation in exponential form

a) $\log_4 64 = 3$

$$4^3 = 64$$

b) $y = \log x$

$$10^y = x$$

Note: because there is no base written, this is understood to be the common logarithm of x .

Part 7: Evaluate a Logarithm

Example 6: Evaluate each logarithm without a calculator

Note: either of the two rules below is appropriate for evaluating logarithmic expressions.

Rule: if $x^a = x^b$, then $a = b$

a) $y = \log_3 81$

$$3^y = 81$$

$$3^y = 3^4$$

$$y = 4$$

b) $y = \log\left(\frac{1}{100}\right)$

$$\begin{aligned}10^y &= \frac{1}{100} \\10^y &= \left(\frac{1}{10}\right)^2 \\10^y &= 10^{-2} \\y &= -2\end{aligned}$$

c) $y = \log_2\left(\frac{1}{8}\right)$

$$\begin{aligned}y &= \log_2\left(\frac{1}{2}\right)^3 \\y &= \log_2 2^{-3} \\y &= -3\end{aligned}$$

Rule: $\log_a(a^b) = b$

d) $y = \log_4 64$

$$\begin{aligned}y &= \log_4(4^3) \\y &= 3\end{aligned}$$