

1.3 Factored Form Polynomial Functions Lesson with solutions

This document was composed and formatted by Annie Bernatchez.

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In this section, you will investigate the relationship between the factored form of a polynomial function and the x -intercepts of the corresponding graph, and you will examine the effect of repeated factors on the graph of a polynomial function.

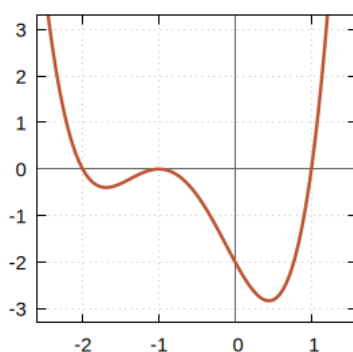
Factored Form Investigation

If we want to graph the polynomial function $f(x) = x^4 + 3x^3 + x^2 - 3x - 2$ accurately, it would be most useful to look at the factored form version of the function:

$$f(x) = (x + 1)^2(x + 2)(x - 1)$$

Let's start by looking at the graph of the function and making connections to the factored form equation.

Graph of $f(x)$:



From the graph, answer the following questions...

- What is the degree of the function? **4**
- What is the sign of the leading coefficient? **Positive**
- What are the x -intercepts? **-2, -1, 1**
- What is the y -intercept? **-2**
- The x -intercepts divide the graph into four intervals. Write the intervals in the first row of the table. In the second row, choose a test point within the interval. In the third row, indicate whether the function is positive (above the x -axis) or negative (below the x -axis).

Interval	$x < -2$	$-2 < x < -1$	$-1 < x < 1$	$x > 1$
Test Point	-3	-1.5	0	2
Sign of $f(x)$	Positive	Negative	Negative	Positive

- What happens to the sign of $f(x)$ near each x -intercept?

At $x = -2$ (order 1) and $x = 1$ (order 1), the sign **changes**. At $x = -1$ (order 2), the sign **does not change** – $f(x)$ is negative on both sides.

Conclusions from investigation:

The x -intercepts of the graph of the function correspond to the roots (zeros) of the corresponding equation. For example, the function $f(x) = (x - 2)(x + 1)$ has x -intercepts at **2** and **-1**. These are the roots of the equation $(x - 2)(x + 1) = 0$.

1.3 Factored Form Polynomial Functions Lesson with solutions

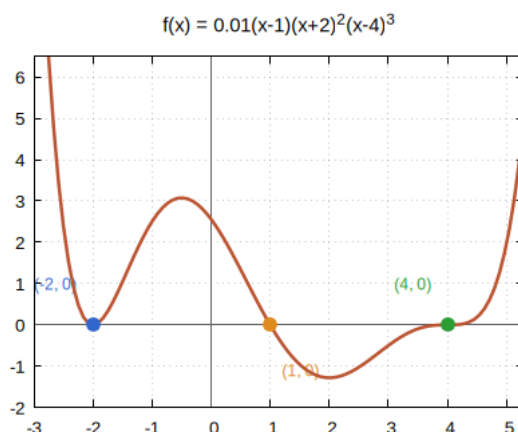
If a polynomial function has a factor $(x - a)$ that is repeated n times, then $x = a$ is a zero of **order** n .

Order – the exponent to which each factor in an algebraic expression is raised.

For example, the function $f(x) = (x - 3)^2(x - 1)$ has a zero of order **2** at $x = 3$ and a zero of order **1** at $x = 1$.

The graph of a polynomial function changes sign at zeros of **odd** order but does not change sign at zeros of **even** order.

Shapes based on order of zero:

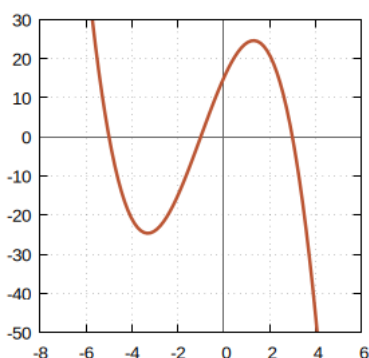


	 Order 2	 Order 1	 Order 3
	$(-2, 0)$ is an x -intercept of order 2. Therefore, it doesn't change sign. "Bounces off" the x -axis. Parabolic shape.	$(1, 0)$ is an x -intercept of order 1. Therefore, it changes sign. "Goes straight through" the x -axis. Linear shape.	$(4, 0)$ is an x -intercept of order 3. Therefore, it changes sign. "S-shape" through the x -axis. Cubic shape.

Example 1: Analyzing Graphs of Polynomial Functions

For each graph,

- the least possible degree and the sign of the leading coefficient
 - the x -intercepts and the factors of the function
 - the intervals where the function is positive/negative
- a)



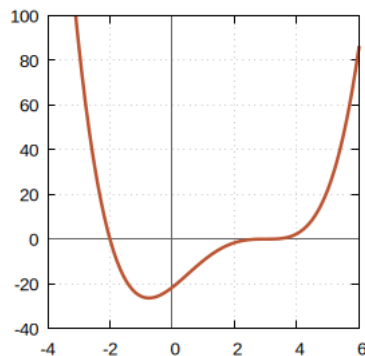
- Least possible degree: **3**. Sign of leading coefficient: **negative**.
- x -intercepts: **-5, -1, 3**. Factors: **$(x+5)$, $(x+1)$, $(x-3)$** – so $f(x) = -a(x+5)(x+1)(x-3)$ for some positive a .
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Interval	$x < -5$	$-5 < x < -1$	$-1 < x < 3$	$x > 3$
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1.3 Factored Form Polynomial Functions Lesson with solutions

Sign of $f(x)$	Positive	Negative	Positive	Negative
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b)



i) Least possible degree: **4**. Sign of leading coefficient: **positive**.

ii) x -intercepts: **-2 (order 1), 3 (order 3)**. Factors: **$(x+2), (x-3)^3$** – so $f(x) = a(x+2)(x-3)^3$ for some positive a .

iii)

Interval	$x < -2$	$-2 < x < 3$	$x > 3$
Sign of $f(x)$	Positive	Negative	Positive

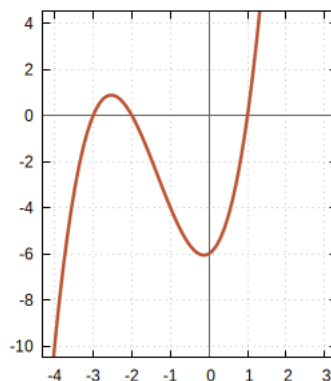
Example 2: Analyze Factored Form Equations to Sketch Graphs

Degree	Leading Coefficient	End Behaviour	x -intercepts	y -intercept
The exponent on x when all factors of x are multiplied together OR Add the exponents on the factors that include an x .	The product of all the x coefficients	Use degree and sign of leading coefficient to determine this	Set each factor equal to zero and solve for x	Set $x = 0$ and solve for y

Sketch a graph of each polynomial function:

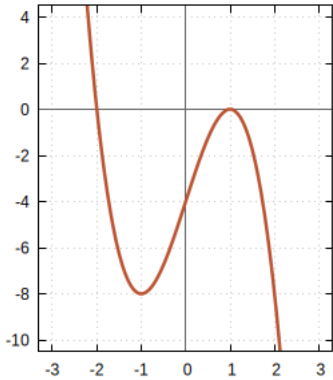
a) $f(x) = (x-1)(x+2)(x+3)$

Degree	Leading Coefficient	End Behaviour	x -intercepts	y -intercept
3	1	$y \rightarrow -\infty$ as $x \rightarrow -\infty$; $y \rightarrow \infty$ as $x \rightarrow \infty$ (Q3 to Q1)	1, -2, -3	-6



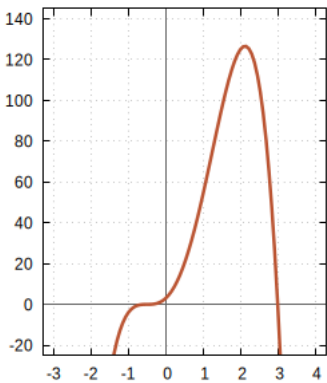
b) $g(x) = -2(x - 1)^2(x + 2)$

Degree	Leading Coefficient	End Behaviour	x-intercepts	y-intercept
3	-2	$y \rightarrow \infty$ as $x \rightarrow -\infty$; $y \rightarrow -\infty$ as $x \rightarrow \infty$ (Q2 to Q4)	1 (order 2), -2 (order 1)	-4



c) $h(x) = -(2x + 1)^3(x - 3)$

Degree	Leading Coefficient	End Behaviour	x-intercepts	y-intercept
4	-8	$y \rightarrow -\infty$ as $x \rightarrow \pm\infty$ (Q3 to Q4)	-0.5 (order 3), 3 (order 1)	3



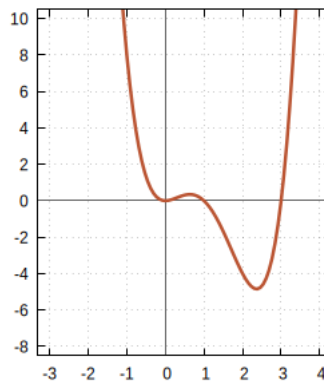
1.3 Factored Form Polynomial Functions Lesson with solutions

d) $j(x) = x^4 - 4x^3 + 3x^2$

Note: must put into factored form to find x -intercepts

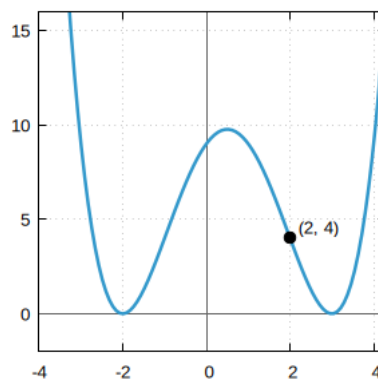
Factored form: $j(x) = x^2(x - 1)(x - 3)$

Degree	Leading Coefficient	End Behaviour	x -intercepts	y -intercept
4	1	$y \rightarrow \infty$ as $x \rightarrow \pm \infty$ (Q2 to Q1)	0 (order 2), 1 (order 1), 3 (order 1)	0



Example 3: Representing the Graph of a Polynomial Function with its Equation

a) Write the equation of the function shown below:



Steps:

1. Write the equation of the family of polynomials using factors created from x -intercepts
2. Substitute the coordinates of another point (x, y) into the equation.
3. Solve for a
4. Write the equation in factored form

The graph touches (doesn't cross) the x -axis at $x = -2$ and $x = 3$, so both are order-2 zeros:

$$f(x) = a(x + 2)^2(x - 3)^2$$

Substituting the point $(2, 4)$:

$$4 = a(2 + 2)^2(2 - 3)^2 = a(16)(1) = 16a \Rightarrow a = \frac{1}{16}$$

$$f(x) = \frac{1}{16}(x + 2)^2(x - 3)^2$$

b) Find the equation of a polynomial function that is degree 4 with zeros -1 (order 3) and 1 , and with a y -intercept of -2 .

$$f(x) = a(x + 1)^3(x - 1)$$

Substituting the y -intercept, $f(0) = -2$:

$$-2 = a(0 + 1)^3(0 - 1) = a(1)(-1) = -a \Rightarrow a = 2$$

$$f(x) = 2(x + 1)^3(x - 1)$$