

1.2 Characteristics of Polynomial Functions Lesson with solutions

This document was composed and formatted by Annie Bernatchez.

The course material originated from www.jensenmath.ca

In section 1.1 we looked at polynomial power functions, which are single-term polynomial functions. Many polynomial functions are made up of two or more terms. In this section we will look at the characteristics of the graphs and equations of polynomial functions.

New Terminology – Local Min/Max vs. Absolute Min/Max

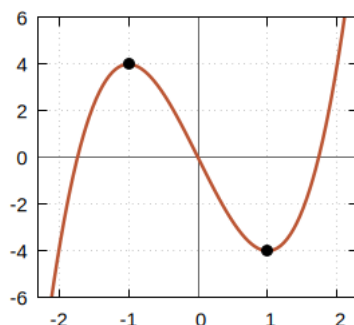
- **Local Min or Max Point** – Points that are minimum or maximum points on some interval around that point.
- **Absolute Max or Min** – The greatest/least value attained by a function for ALL values in its domain.

Investigation: Graphs of Polynomial Functions

The degree and the leading coefficient in the equation of a polynomial function indicate the end behaviours of the graph.

The degree of a polynomial function provides information about the shape, turning points (local min/max), and zeros (x-intercepts) of the graph.

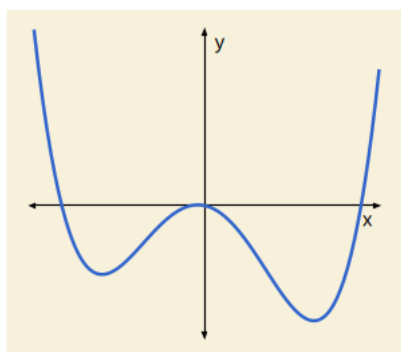
In the graph below, $(-1, 4)$ is a **local maximum** and $(1, -4)$ is a **local minimum**. These are not absolute min and max points because there are other points on the graph of the function that are smaller and greater. Sometimes local min and max points are called **turning points**.



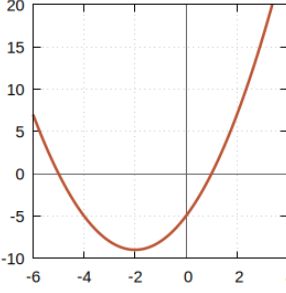
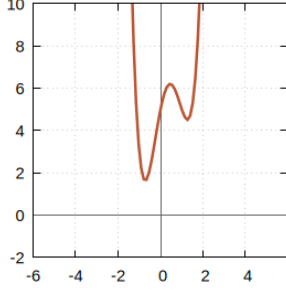
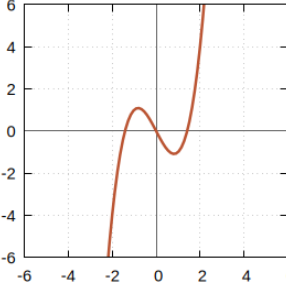
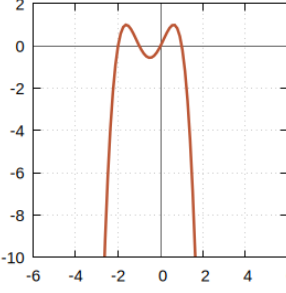
On the graph below...

There are **3** local min/max points. **2** are local min and **1** is a local max.

One of the local min points is also an absolute min (the one on the right).



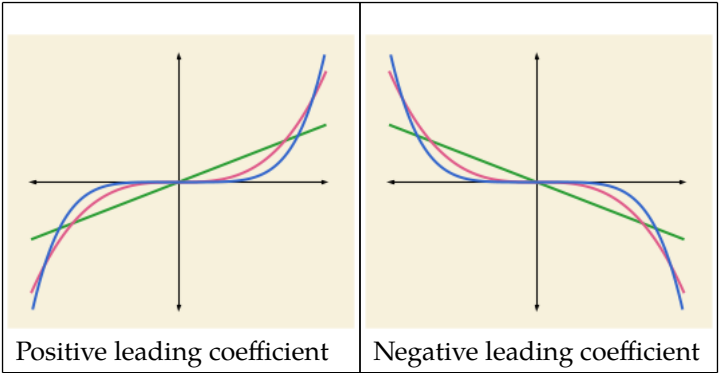
Complete the following table using the equation and graphs given:

Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviour	Number of turning points	Number of x-intercepts
$f(x) = x^2 + 4x - 5$ 	2	Even	1	$y \rightarrow \infty$ as $x \rightarrow \pm \infty$ (Q2 to Q1)	1	2
$f(x) = 3x^4 - 4x^3 - 4x^2 + 5x + 5$ 	4	Even	3	$y \rightarrow \infty$ as $x \rightarrow \pm \infty$ (Q2 to Q1)	3	0
$f(x) = x^3 - 2x$ 	3	Odd	1	$y \rightarrow -\infty$ as $x \rightarrow -\infty$; $y \rightarrow \infty$ as $x \rightarrow \infty$ (Q3 to Q1)	2	3
$f(x) = -x^4 - 2x^3 + x^2 + 2x$ 	4	Even	-1	$y \rightarrow -\infty$ as $x \rightarrow \pm \infty$ (Q3 to Q4)	3	4

$f(x) = 2x^6 - 12x^4 + 18x^2 + x - 10$	6	Even	2	$y \rightarrow \infty$ as $x \rightarrow \pm \infty$ (Q2 to Q1)	5	2
$f(x) = 2x^5 + 7x^4 - 3x^3 - 18x^2 + 5$	5	Odd	2	$y \rightarrow -\infty$ as $x \rightarrow -\infty$; $y \rightarrow \infty$ as $x \rightarrow \infty$ (Q3 to Q1)	4	5

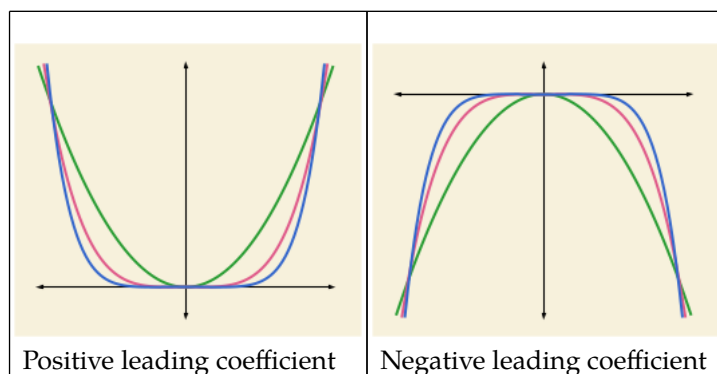
Summary of Findings:

- A polynomial function of degree n has at most $n - 1$ local max/min points (turning points)
- A polynomial function of degree n may have up to n distinct zeros (x-intercepts)
- If a polynomial function is **odd** degree, it must have at least one x-intercept, and an even number of turning points
- If a polynomial function is **even** degree, it may have no x-intercepts, and an odd number of turning points
- An odd degree polynomial function extends from...
 - **Q3** quadrant to **Q1** quadrant if it has a positive leading coefficient
 - **Q2** quadrant to **Q4** quadrant if it has a negative leading coefficient



Note: Odd degree polynomials have OPPOSITE end behaviours

- An even degree polynomial function extends from...
 - Q2 quadrant to Q1 quadrant if it has a positive leading coefficient
 - Q3 quadrant to Q4 quadrant if it has a negative leading coefficient



Note: Even degree polynomials have THE SAME end behaviour

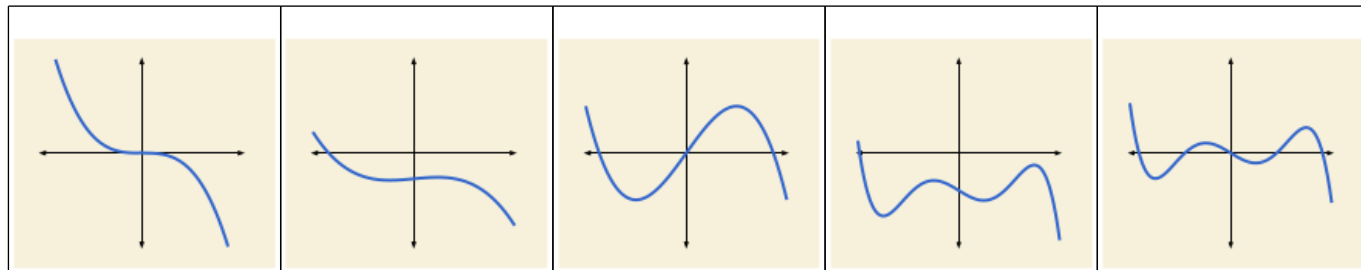
Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviour	Number of turning points	Number of x-intercepts
$f(x) = 5x^5 + 5x^4 - 2x^3 + 4x^2 - 3x$	5	Odd	5	$y \rightarrow -\infty$ as $x \rightarrow -\infty$; $y \rightarrow \infty$ as $x \rightarrow \infty$ (Q3 to Q1)	2	3
$f(x) = -2x^3 + 4x^2 - 3x - 1$	3	Odd	-2	$y \rightarrow \infty$ as $x \rightarrow -\infty$; $y \rightarrow -\infty$ as $x \rightarrow \infty$ (Q2 to Q4)	0	1
$f(x) = x^4 + 2x^3 - 3x - 1$	4	Even	1	$y \rightarrow \infty$ as $x \rightarrow \pm\infty$ (Q2 to Q1)	1	2

Example 1: Describe the end behaviours of each function, the possible number of turning points, and the possible number of zeros. Use these characteristics to sketch possible graphs of the function

a) $f(x) = -3x^5 + 4x^3 - 8x^2 + 7x - 5$

Degree 5 (odd), leading coefficient -3 (negative). End behaviour: $y \rightarrow \infty$ as $x \rightarrow -\infty$; $y \rightarrow -\infty$ as $x \rightarrow \infty$ (extends from Q2 to Q4). Possible number of turning points: 0, 2, or 4. Possible number of zeros: 1, 3, or 5.

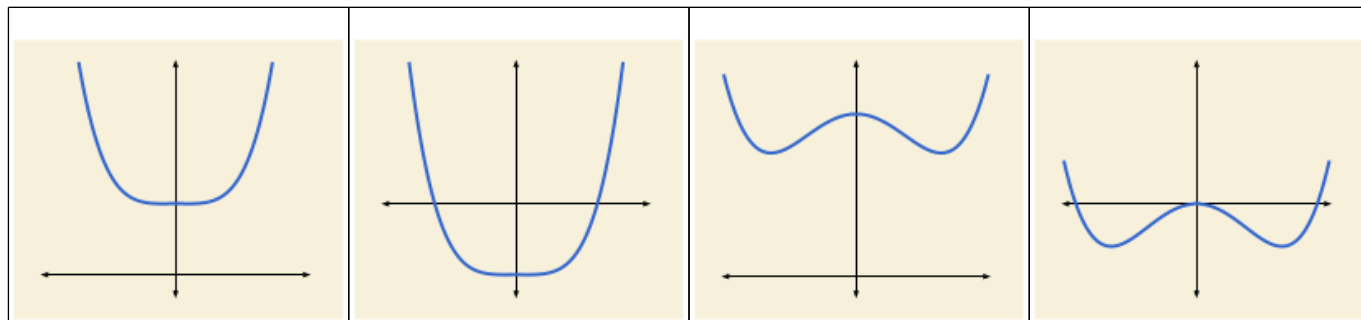
Possible graphs of 5th degree polynomial functions with a negative leading coefficient:



b) $g(x) = 2x^4 + x^2 + 2$

Degree 4 (even), leading coefficient 2 (positive). End behaviour: $y \rightarrow \infty$ as $x \rightarrow \pm \infty$ (extends from Q2 to Q1). Possible number of turning points: 1 or 3. Possible number of zeros: 0, 2, or 4.

Possible graphs of 4th degree polynomial functions with a positive leading coefficient:



Example 2: Fill out the following chart

Degree	Possible # of x -intercepts	Possible # of turning points
1	1	0
2	2	1
3	3	2
4	4	3
5	5	4

Note: Odd degree functions must have an even number of turning points.

Note: Even degree functions must have an odd number of turning points.

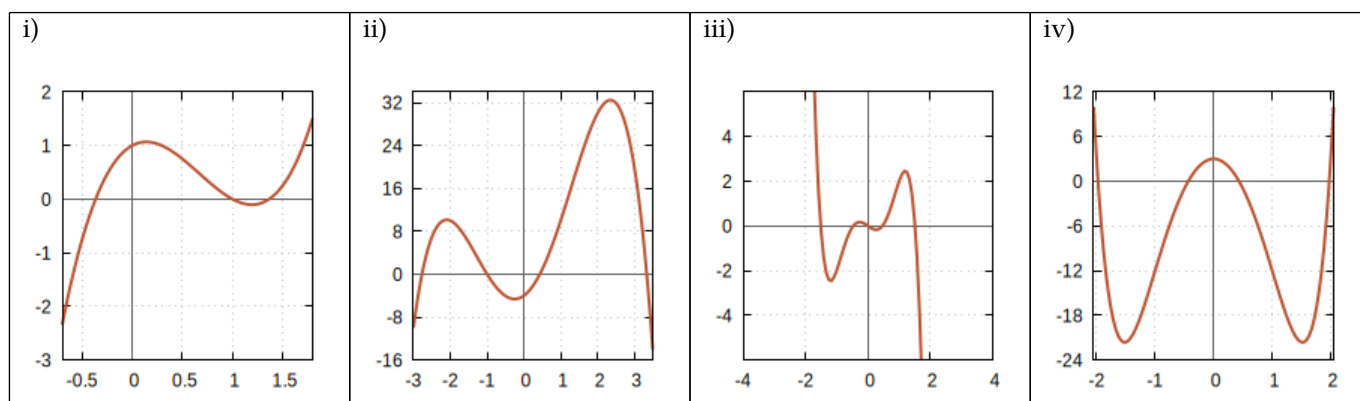
Example 3: Determine the key features of the graph of each polynomial function. Use these features to match each function with its graph. State the number of x -intercepts, the number of local max/min points, and the number of absolute max/min points for the graph of each function. How are these features related to the degree of each function?

a) $f(x) = 2x^3 - 4x^2 + x + 1$

b) $g(x) = -x^4 + 10x^2 + 5x - 4$

c) $h(x) = -2x^5 + 5x^3 - x$

d) $p(x) = x^6 - 16x^2 + 3$



- a) Matches i). 3 x-intercepts, 2 local max/min points (1 local max, 1 local min), 0 absolute max/min points. Odd degree (3): unbounded in both directions, so no absolute extrema.
- b) Matches ii). 4 x-intercepts, 3 local max/min points (2 local max, 1 local min), 1 absolute max point (the larger of the two local maxima; no absolute min). Even degree (4) with a negative leading coefficient: bounded above, giving an absolute maximum.
- c) Matches iii). 5 x-intercepts, 4 local max/min points, 0 absolute max/min points. Odd degree (5): unbounded in both directions, so no absolute extrema.
- d) Matches iv). 4 x-intercepts, 3 local max/min points (1 local max, 2 local min), 2 absolute min points (the two local minima are tied, by symmetry; no absolute max). Even degree (6) with a positive leading coefficient: bounded below, giving an absolute minimum.

In general: the maximum possible number of turning points is one less than the degree, and the maximum possible number of x-intercepts equals the degree. Absolute extrema only occur for even-degree functions, since only those are bounded in one direction (odd-degree functions always extend to both $+\infty$ and $-\infty$).

Finite Differences

For a polynomial function of degree n , where n is a positive integer, the n^{th} differences...

- are equal
- have the same sign as the leading coefficient
- are equal to $a \cdot n!$, where a is the leading coefficient

Note:

- $n!$ is read as n factorial.
- $n! = n \times (n-1) \times (n-2) \times \dots \times 2 \times 1$
- $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Example 4: The table of values represents a polynomial function. Use finite differences to determine

- the degree of the polynomial function
- the sign of the leading coefficient
- the value of the leading coefficient

x	y	First differences	Second differences	Third differences
-3	-36	N/A	N/A	N/A
-2	-12	24	N/A	N/A
-1	-2	10	-14	N/A
0	0	2	-8	6
1	0	0	-2	6
2	4	4	4	6

3	18	14	10	6
4	48	30	16	6

- a) The third differences are constant, so the degree is 3.
b) The constant differences (6) are positive, so the leading coefficient is positive.
c) Constant third differences $= a \cdot 3! = 6a = 6$, so $a = 1$

Example 5: For the function $2x^4 - 4x^2 + x + 1$ what is the value of the constant finite differences?

Finite differences $= a \cdot n! = 2 \cdot 4! = 2 \times 24 = 48$